Représentations typiques pour GL_n(F)

Santosh Vrn Nadimpalli

THÈSE DE DOCTORAT

Présentée pour obtenir

LE GRADE DE DOCTEUR EN SCIENCES
DE L'UNIVERSITÉ PARIS-SUD

Préparée à l'École Doctorale de Mathématiques
de la région Paris-Sud (ED 142)

Cotutelle par Mathematisch Instituut,
Université de Leiden (Pays-Bas)

Spécialité : Mathématiques

par

Santosh Nadimpalli

Typical representations for $\mathrm{GL}_n(F)$

Soutenue le 16 Juin 2015 devant la Commission d'examen :

Mme. Corinne	Blondel	Université Paris Diderot - Paris 7	Rapporteur
M. Bas	Edixhoven	Universiteit Leiden	Directeur de thèse
M. David	Helm	Imperial College, London	Rapporteur (excusé)
M. Guy	Henniart	Université Paris-Sud	Directeur de thèse
M. Hendrik	Lenstra	Universiteit Leiden	Examinateur
M. Peter	Stevenhagen	Universiteit Leiden	Examinateur
M. Shaun	Stevens	University of East Anglia	Examinateur

Thèse préparée au
Département de Mathématiques d'Orsay
Laboratoire de Mathématiques (UMR 8628), Bât. 425
Université Paris-Sud
91405 Orsay CEDEX

Contents

Curriculum Vitae **140**

Résumé des résultats

Soit F un corps local commutatif à corps résiduel fini k_F. Dans cette thèse nous étudions, pour $n \geq 2$, la restriction d'une représentation lisse irréductible de $\mathrm{GL}_n(F)$ à un sous-groupe compact maximal K. En particulier, nous nous intéressons aux représentations irréductibles lisses (appelées représentations K-typiques) de K qui déterminent le support inertiel de la représentation lisse irréductible donnée. Dans le contexte de la correspondance de Langlands locale, ces représentations ont eu des applications arithmétiques importantes. Dans cette thèse, nous essayons de réaliser, dans de nombreux cas, la classification de ces représentations lisses irréductibles de K pour un support inertiel donné s.

Motivation

Correspondance de Langlands locale pour GL_n

Nous fixons une clôture algébrique séparable $\bar{F}$ de F. Soit F^{un} la sous-extension maximale non-ramifiée dans $\bar{F}$. Nous avons une application canonique

$$\mathrm{Gal}(\bar{F}/F) \to \mathrm{Gal}(F^{un}/F).$$

Le groupe $\mathrm{Gal}(F^{un}/F)$ est canoniquement isomorphe au groupe de Galois du corps résiduel $k_{F^{un}}$ de F^{un} sur k_F. Comme $k_{F^{un}}$ est la clôture algébrique du corps fini k_F nous obtenons l'application

$$\mathrm{Gal}(\bar{F}/F) \to \mathrm{Gal}(k_{F^{un}}/k_F) \simeq \hat{\mathbb{Z}}. \tag{1}$$

Soit q le cardinal du corps résiduel k_F. On note par Φ_F l'automorphisme de $k_{F^{un}}$ qui envoie un élément x à x^q. Soit W_F le groupe constitué par les éléments de $\mathrm{Gal}(\bar{F}/F)$ qui induisent une puissance entière de Φ_F par l'application (1). Le groupe W_F est appelé le groupe de Weil de F. Le groupe W_F devient un groupe localement compact en déclarant les sous-groupes ouverts de $\mathrm{Gal}(\bar{F}/F^{un})$ (avec sa topologie profinie) sous-groupes ouverts de W_F. Ainsi nous obtenons une suite exacte de groupes topologiques

$$0 \to \mathrm{Gal}(\bar{F}/F^{un}) \to W_F \to \Phi_F^{\mathbb{Z}} \to 0$$

où $\Phi_F^{\mathbb{Z}}$ est muni de la topologie discrète.

La théorie des corps de classes locaux nous donne un isomorphisme topologique

$$W_F^{ab} \simeq F^{\times},$$

où W_F^{ab} est le quotient de W_F par l'adhérence du groupe dérivé de W_F. Cela nous donne une bijection entre les caractères continus de W_F et ceux de

$F^\times$. La correspondance de Langlands locale établit un analogue en dimension supérieure de la correspondance entre les caractères obtenus par la théorie du corps de classes local. Une telle correspondance peut être formulée en termes de certain objets algébriques appelés représentations de Weil-Deligne. Pour commencer, nous introduisons une norme $\|\|$ sur le groupe de Weil W_F. Soit x un élément de W_F, d'image Φ_F^r par la application (1), alors $\|x\| = q^{-r}$. Une représentation de Weil-Deligne de dimension n est un triplet (r, V, N) où V est un espace vectoriel complexe de dimension n, r est un homomorphisme de W_F dans $\mathrm{GL}(V)$ dont le noyau est ouvert, N est un élément de $\mathrm{End}_{\mathbb{C}}(V)$ qui satisfait

$$r(x)Nr(x)^{-1} = \|x\|N.$$

pour tout $x \in W_F$. Nous disons qu'un triplet (r, V, N) est Frobenius semi-simple si la représentation (r, V) est semi-simple. La correspondance de Langlands locale (voir [LRS93] pour le cas où la caractéristique de F est non nulle et [HT01] et [Hen00] pour le cas où la caractéristique de F est égal à zéro) est une correspondance naturelle entre l'ensemble des classes isomorphismes de représentation Weil-Deligne de dimension n, Frobenius semi-simples, et l'ensemble des classes d'isomorphisme de représentations irréductibles complexes lisses de $\mathrm{GL}_n(F)$. (Une représentation (π, V) est dite lisse si et seulement si pour tout $v \in V$ le stabilisateur de v est ouvert dans $\mathrm{GL}_n(F)$ pour la topologie induite à partir de F).

Soit B_n l'ensemble des couples (M, σ) où M est un sous-groupe de Levi d'un sous-groupe parabolique de $\mathrm{GL}_n(F)$ et σ une représentation irréductible cuspidale de M. Nous rappelons que **l'équivalence inertielle** est une relation d'équivalence sur l'ensemble B_n définie par $(M_1, \sigma_1) \sim (M_2, \sigma_2)$ si et seulement s'il existe un élément $g \in G$ et un caractère non ramifié χ de M_2 tel que $M_1 = gM_2g^{-1}$ et $\sigma_1^g \simeq \sigma_2 \otimes \chi$. Nous utilisons la notation $[M, \sigma]$ pour la classe d'équivalence contenant le couple (M, σ). Les classes d'équivalence sont également appelées **classes inertielles**. Chaque représentation irréductible lisse intervient dans une représentation induite parabolique $i_P^{\mathrm{GL}_n(F)}(\sigma)$ où σ est une représentation irréductible cuspidale d'un sous-groupe de Levi M de P. Le couple (M, σ) est bien déterminé à $\mathrm{GL}_n(F)$-conjugaison près (voir [BZ77][Theorem 2.5 Theorem 2.9(a)(i)]). La classe $[M, \sigma]$ est appelé **le support inertiel** de π.

Étant donné deux triples (r_1, V_1, N_1) et (r_2, V_2, N_2), il se trouve que les restrictions de r_1 et r_2 au groupe $\mathrm{Gal}(\bar{F}/F^{un})$ sont isomorphes si et seulement si les représentations lisses π_1 et π_2 associées par la correspondance de Langlands locale pour (r_1, V_1, N_1) et (r_2, V_2, N_2) respectivement ont le même support inertiel. Dans plusieurs applications arithmétiques (par exemple voir [BM02] et [EG14]) on cherche à associer à un support inertiel donné, disons s, une représentation lisse irréductible τ de $\mathrm{GL}_n(\mathcal{O}_F)$ qui a la propriété que si $\mathrm{Hom}_{\mathrm{GL}_n(\mathcal{O}_F)}(\tau, \pi) \neq 0$ alors le support inertiel de π est s. Une telle représentation est appelée $\mathrm{GL}_n(\mathcal{O}_F)$**-représentation typique**.

Théorie des types

Soit G le groupe de F-points d'un groupe réductif algébrique. Il a été montré par Bernstein que la catégorie des représentations lisses $\mathcal{M}(G)$ admet une décomposition

$$\mathcal{M}(G) = \prod_{s \in \mathcal{B}(G)} \mathcal{M}_s(G)$$

où $\mathcal{M}_s(G)$ est la sous-catégorie pleine composée des représentations lisses telles que tous leurs sous-quotients irréductibles ont support inertiel s. La théorie des types développée initialement par Bushnell-Kutzko (voir [BK98] pour une discussion générale sur les représentations lisses et la théorie des types) donne une construction de couples (J_s, λ_s) où J_s est un sous-groupe ouvert compact de G et λ_s est une représentation irréductible lisse de J_s telle que $\mathrm{Hom}_{J_s}(\lambda_s, \pi) \neq 0$ si et seulement si $\pi \in \mathcal{M}_s(G)$ pour toutes les représentations lisses irréductibles π de G. Un tel couple (J_s, λ_s) est appelé un **type** pour s. Un tel type (J_s, λ_s) donne une équivalence naturelle de catégories entre $\mathcal{M}_s(G)$ et la catégorie des modules sur l'algèbre de Hecke sphérique $\mathcal{H}(J_s, \lambda_s)$.

Soit K un sous-groupe compact maximal de G et s une classe inertielle de G. Soit (J_s, λ_s) un type pour s telle que $J_s \subset K$. Soit Γ une sous-représentation irréductible de

$$\mathrm{ind}_{J_s}^{K}(\lambda_s). \tag{2}$$

Soit π une représentation lisse irréductible de G telle que $\mathrm{res}_K \pi$ contient Γ. Alors $\mathrm{res}_{J_s} \pi$ contient λ_s par réciprocité de Frobenius. Donc le support inertiel de π est s. Cela montre que les sous-représentations irréductibles de (2) sont K-typiques. Comme l'existence de types n'est pas connue dans tous les cas, les questions naturelles suivantes se posent:

1. Est-ce qu'une représentation K-typique existe?
2. Pour une classe inertielle s, est-ce que les représentations K-typiques sont en nombre fini?
3. Comment classifier les représentations K-typiques?

Pour $G = \mathrm{GL}_n(F)$ des types (J_s, λ_s) ont été explicitement construits par Bushnell-Kutzko dans les articles [BK93] et [BK99]. Pour $\mathrm{GL}_n(F)$, nous choisissons "le type de Bushnell-Kutzko" (J_s, λ_s) tel que $J_s \subset \mathrm{GL}_n(\mathcal{O}_F)$ pour tout $s \in \mathcal{B}_n$. Dans cette thèse, nous donnons des réponses aux questions ci-dessus pour $G = \mathrm{GL}_n(F)$ en termes de la théorie des types. Pour $\#k_F > 3$ nous montrons dans de nombreux cas que les sous-représentations irréductibles de

$$\mathrm{ind}_{J_s}^{\mathrm{GL}_n(\mathcal{O}_F)}(\lambda_s)$$

sont précisément les représentations typiques pour la classe inertielle s. En ce sens, nous classifions les représentations typiques pour la classe s. Notons que les types construits par Bushnell-Kutzko ne sont pas toujours uniques, même à conjugaison près. Nous utilisons la terminologie “le type de Bushnell-Kutzko” pour le couple (J_s, λ_s), le type pour $s = [M, \sigma]$ construit par la procédure inductive dans l'article [BK99] après la fixation d'un type pour la classe inertielle $[M, \sigma]$ de M.

Il existe différentes méthodes de construction de types (J, λ) pour une classe inertielle donnée s (en ce sens que le couple (J, λ) a la propriété $\mathrm{Hom}_J(\lambda, \pi) \neq 0$ si et seulement si le support inertiel de π est s pour toute représentation lisse irréductible π de G). Pour une telle construction et K un sous-groupe compact maximal contenant J, les sous-représentations irréductibles de

$$\mathrm{ind}_J^K(\lambda)$$

sont des représentations K-typiques pour s. La théorie des représentations typiques, au moins pour le cas de GL_n, vise à donner une approche uniforme. Il pourrait être intéressant de prouver au moins la finitude du nombre des représentations K-typiques dans le cas général.

Résultats connus

Le cas de $\mathrm{GL}_2(F)$ est traité par Henniart dans l'annexe à l'article [BM02]. Il a complètement classifié les représentations typiques pour toutes les classes inertielles. Henniart prédit que ses résultats peuvent être étendus à $\mathrm{GL}_n(F)$. Paskunas a classé les représentations typiques pour les classes inertielles $[\mathrm{GL}_n(F), \sigma]$. Nous décrivons maintenant les résultats d'Henniart et de Paskunas. Nous remarquons que J_s peut être conjugué à un sous-groupe de $\mathrm{GL}_n(\mathcal{O}_F)$. Nous supposerons que J_s est un sous-groupe de $\mathrm{GL}_n(\mathcal{O}_F)$.

Théorème 0.0.1 (Henniart)**.** *Soit s une classe inertielle pour $\mathrm{GL}_2(F)$. Soit (J_s, λ_s) le type de Bushnell-Kutzko pour la classe inertielle s. Si $\#k_F > 2$ et $s = [\mathrm{GL}_2(F), \sigma]$ alors les représentations typiques de s sont les sous-représentations irréductibles de*

$$\mathrm{ind}_{J_s}^{\mathrm{GL}_2(\mathcal{O}_F)}(\lambda_s).$$

Soit T le tore maximal de $\mathrm{GL}_2(F)$ constitué des matrices diagonales et soit $s = [T, \chi]$ une classe inertielle pour $\mathrm{GL}_2(F)$. Nous identifions T à $F^\times \times F^\times$ et le caractère χ à $\chi((a, b)) = \chi_1(a)\chi_2(b)$ pour deux caractères χ_1 et χ_2 de $F^\times$. Soit B un sous-groupe de Borel contenant T. Soit $B(m)$ le groupe des matrices de $\mathrm{GL}_2(\mathcal{O}_F)$ qui, sous la réduction mod $\mathfrak{P}_F^m$ se trouvent dans le groupe $B(\mathcal{O}_F/\mathfrak{P}_F^m)$. Soit N le niveau de $\chi_1\chi_2^{-1}$. Si $\chi_1\chi_2^{-1} \neq \mathrm{id}$, et si $\#k_F = 2$, Henniart a montré que

$$\mathrm{ind}_{B(N)}^{\mathrm{GL}_2(\mathcal{O}_F)}(\chi) \quad \text{et} \quad \mathrm{ind}_{B(N+1)}^{\mathrm{GL}_2(\mathcal{O}_F)}(\chi)$$

sont typiques pour la classe inertielle s. Le type de Bushnell-Kutzko pour la classe inertielle $s = [T, \chi]$ est donné par $(B(N), \chi)$. Donc, cela montre que dans le cas présent il y a en effet des représentations supplémentaires autres que les sous-représentations irréductibles de (2) qui sont typiques pour s. Pour toutes les autres classes inertielles $[T, \chi]$ Henniart montre que les représentations typiques sont des sous-représentations de (2).

Pour la classe inertielle $s = [\mathrm{GL}_n(F), \sigma]$ il découle facilement que

$$\mathrm{ind}_{J_s}^{\mathrm{GL}_n(\mathcal{O}_F)}(\lambda_s)$$

est une représentation irréductible et Paskunas dans l'article [Pas05] a montré le théorème suivant.

Théorème 0.0.2 (Paskunas). *Pour tout entier $n > 1$ et pour toute classe inertielle $s = [\mathrm{GL}_n(F), \sigma]$ il existe une seule représentation typique pour s.*

Résultats de cette thèse

Dans cette thèse, nous nous intéressons à la classification des représentations typiques pour les classes inertielles $[M, \sigma]$ où M est un sous-groupe de Levi propre de $\mathrm{GL}_n(F)$ et $n \geq 3$. Alors nous choisissons le type de Bushnell-Kutzko (J_s, λ_s) est conjugué de sorte que $J_s \subset \mathrm{GL}_n(\mathcal{O}_F)$. Nous donnons une description détaillée des résultats de chaque chapitre.

Résultats du chapitre 2

Si τ est une représentation typique pour une classe inertielle s alors nous montrons que τ est un sous-représentation d'une représentation irréductible lisse notée $\pi \in \mathcal{M}_s(G)$ de $\mathrm{GL}_n(F)$. Nous choisissons un représentant (M, σ) de s tel que M est le sous-groupe de Levi composé de matrices diagonales par blocs et σ naturellement une représentation supercuspidale de M. Maintenant, la représentation π est une sous-représentation de

$$i_P^{\mathrm{GL}_n(F)}(\sigma \otimes \chi)$$

où P est un sous-groupe parabolique contenant M comme sous-groupe de Lévi et $i_P^{\mathrm{GL}_n(F)}$ est l'induction parabolique et χ est un caractère non ramifié de M. Ainsi, pour la classification des représentations typiques, il faut trouver les représentations typiques qui apparaissent dans la représentation

$$\mathrm{res}_{\mathrm{GL}_n(\mathcal{O}_F)}\{i_P^{\mathrm{GL}_n(F)}(\sigma)\} \simeq \mathrm{ind}_{P\cap\mathrm{GL}_n(\mathcal{O}_F)}^{\mathrm{GL}_n(\mathcal{O}_F)}(\sigma). \tag{3}$$

Maintenant, nous identifions M au produit

$$\mathrm{GL}_{n_1}(F) \times \mathrm{GL}_{n_2}(F) \times \dots \mathrm{GL}_{n_r}(F)$$

pour une partition ordonnée $(n_1, n_2, \dots, n_r)$ de n et σ au produit tensoriel $\sigma_1 \boxtimes \sigma_2 \boxtimes \dots \boxtimes \sigma_r$ où σ_i est une représentation cuspidale de $\mathrm{GL}_{n_i}(F)$. Soit τ_i la représentation typique unique dans σ_i. Il a été montré par Will Conley que la représentation

$$\mathrm{ind}_{P\cap \mathrm{GL}_n(\mathcal{O}_F)}^{\mathrm{GL}_n(\mathcal{O}_F)}(\boxtimes_{i=1}^r \tau_i)$$

admet un complément X dans (3) tel que les $\mathrm{GL}_n(\mathcal{O}_F)$-sous-représentations irréductibles de X sont non-typiques pour la classe inertielle s.

Maintenant, toutes les représentations typiques pour s sont des $\mathrm{GL}_n(\mathcal{O}_F)$ sous-représentations irréductibles de

$$\mathrm{ind}_{P\cap \mathrm{GL}_n(\mathcal{O}_F)}^{\mathrm{GL}_n(\mathcal{O}_F)}(\boxtimes_{i=1}^r \tau_i).$$

Cette représentation est toujours une représentation de dimension infinie. La première idée est de construire des sous-groupes compacts ouverts H_m pour $m \geq 1$ tels que $H_{m+1} \subset H_m$,

$$\cap_{m\geq 1} H_m = P \cap \mathrm{GL}_n(\mathcal{O}_F)$$

et que $\boxtimes_{i=1}^r \tau_i$ se prolonge à une représentation de H_1. Avec quelques conditions supplémentaires, nous montrons que

$$\mathrm{ind}_{P\cap \mathrm{GL}_n(\mathcal{O}_F)}^{\mathrm{GL}_n(\mathcal{O}_F)}(\boxtimes_{i=1}^r \tau_i) \simeq \bigcup_{m\geq 1} \mathrm{ind}_{H_m}^{\mathrm{GL}_n(\mathcal{O}_F)}(\boxtimes_{i=1}^r \tau_i).$$

Pour tout support inertiel nous définissons les H_m dans chaque chapitre et nous analysons les représentations

$$\mathrm{ind}_{H_m}^{\mathrm{GL}_n(\mathcal{O}_F)}(\boxtimes_{i=1}^r \tau_i).$$

En plus de cela, nous montrons plusieurs lemmes qui sont fréquemment utilisés dans l'ensemble de cette thèse.

Résultats du chapitre 3

Ce chapitre concerne les classes inertielles, dites classes inertielles de niveau zéro,

$$[M = \prod_{i=1}^r \mathrm{GL}_{n_i}(F), \boxtimes_{i=1}^r \sigma_i]$$

où chaque σ_i contient un vecteur non nul fixé par le sous-groupe de congruence principal de niveau un. Le type de Bushnell-Kutzko pour σ_i est donné par

$(\mathrm{GL}_{n_i}(\mathcal{O}_F), \tau_i)$ où τ_i est obtenu par inflation d'une représentation cuspidale de $\mathrm{GL}_{n_i}(k_F)$ pour tous $i \leq r$. Maintenant, choisissons P le groupe de matrices diagonales supérieures par blocs contenant M comme un sous-groupe de Levi. Pour un entier $m \geq 1$ nous désignons par $P(m)$ le groupe de matrices dans $\mathrm{GL}_n(\mathcal{O}_F)$ qui, par réduction mod-$\mathfrak{P}_F^m$ appartiennent au groupe $P(\mathcal{O}_F/\mathfrak{P}_F^m)$. Maintenant, la représentation $\boxtimes_{i=1}^r \tau_i$ peut être vue comme une représentation de $P(1)$ par inflation. La suite des groupes H_m, que nous avons annoncée dans la sous-section précédente, est donnée par les $P(m)$.

Pour simplifier la notation, nous noterons V_m la représentation

$$\mathrm{ind}_{P(m)}^{\mathrm{GL}_n(\mathcal{O}_F)}(\boxtimes_{i=1}^r \tau_i).$$

Par récurrence sur l'entier positif m, nous montrons le théorème suivant.

Théorème 0.0.3. *Les $\mathrm{GL}_n(\mathcal{O}_F)$-sous-représentations irréductibles de V_m/V_1 ne sont pas typiques pour la classe inertielle $s = [M, \sigma]$.*

Nous construisons en fait un complément de V_1 dans V_m.

Nous notons que le type de Bushnell-Kutzko pour la classe inertielle s est donnée par le couple $(P(1), \boxtimes_{i=1}^r \tau_i)$. Du théorème ci-dessus, nous pouvons conclure que le théorème ci-dessus classe complètement les représentations typiques dans ce cas. Grâce à notre analyse, nous gagnons quelques informations supplémentaires. Nous concluons le résultat dans le théorème suivant.

Théorème 0.0.4. *Soit $s = [M, \sigma]$ une classe inertielle de niveau zéro. Soit Γ une représentation typique pour la classe inertielle s. La représenaentation Γ est une sous-représentation irréductible de V_1 et*

$$\dim_{\mathbb{C}} \mathrm{Hom}_{\mathrm{GL}_n(\mathcal{O}_F)}(\Gamma, V_1) = \dim_{\mathbb{C}} \mathrm{Hom}_{\mathrm{GL}_n(\mathcal{O}_F)}(\Gamma, i_P^{\mathrm{GL}_n(F)}(\sigma)).$$

Résultats du chapitre 4

Soit T_n le tore maximal composé des matrices diagonales inversibles dans $\mathrm{GL}_n(F)$. Pour $\#k_F > 3$ nous déterminons les représentations typiques pour les classes inertielles $s = [T_n, \chi]$. Nous allons montrer que les représentations typiques apparaissent comme des sous-représentations de

$$\mathrm{ind}_{J_s}^{\mathrm{GL}_n(\mathcal{O}_F)}(\lambda_s).$$

Premièrement, notre but est de définir les groupes H_m tel que $H_{m+1} \subset H_m$ et $\cap_{m \geq 1} H_m = B_n(\mathcal{O}_F)$ où B_n est le sous-groupe de Borel composé de matrices triangulaires supérieures inversibles. Nous notons que le type de Bushnell-Kutzko (J_s, λ_s) dans ce cas, a la propriété $J_s \cap B_n = B_n(\mathcal{O}_F)$. Nous pouvons donc choisir $H_1 = J_s$ et les autres groupes doivent être définies plus précisément. Nous esquissons les détails.

Nous identifions T_n au groupe $\times_{i=1}^n F^\times$ et le caractère χ à $\boxtimes_{i=1}^n \chi_i$ où χ_i est un caractère lisse de $F^\times$. On note $l(\chi)$ l'entier $k > 0$ minimal tel que $1 + \mathfrak{P}_F^k$ est contenu dans le noyau de χ. Soit $J_\chi(m)$ l'ensemble constitué de matrices (a_{ij}) où $a_{ij} \in \mathfrak{P}_F^{l(\chi_i\chi_j^{-1})+m-1}$ pour tout $i > j$, $a_{ii} \in \mathcal{O}_F^\times$ et $a_{ij} \in \mathcal{O}_F$ pour tout $i < j$. Nous allons montrer que $J_\chi(m)$ est en effet un sous-groupe ouvert compact et nous établissons également que

1. $J_s = J_\chi(1)$
2. $\cap_{m\geq 1} J_\chi(m) = B_n(\mathcal{O}_F)$
3. Le caractére χ de $T(\mathcal{O}_F)$ s'étend à un caractère de $J_\chi(1)$.

Par conséquent, nous définissons $H_m = J_\chi(m)$ pour tout $m \geq 1$. Cela montre que

$$\mathrm{res}_{\mathrm{GL}_n(\mathcal{O}_F)}\, i_{B_n}^{\mathrm{GL}_n(F)}(\chi) \simeq \bigcup_{m\geq 1} \mathrm{ind}_{J_\chi(m)}^{\mathrm{GL}_n(\mathcal{O}_F)}(\chi)$$

On note $V_m(\chi)$ la représentation

$$\mathrm{ind}_{J_\chi(m)}^{\mathrm{GL}_n(\mathcal{O}_F)}(\chi).$$

En utilisant une réccurence sur les entiers $n > 0$ et $m > 0$, nous montrons le théorème suivant.

Théorème 0.0.5. *Les $\mathrm{GL}_n(\mathcal{O}_F)$ sous-représentations irréductibles de $V_m(\chi)/V_1(\chi)$ ne sont pas typiques pour les classes inertielles de $s = [T_n, \chi]$.*

Résultats du chapitre 5

Dans ce chapitre, nous nous intéressons à la classification des représentations typiques pour les classes inertielles $s = [\mathrm{GL}_n(F) \times GL_1(F), \sigma \boxtimes \chi]$. Dans ce chapitre et le suivant, nous allons utiliser les techniques de [BK93] comme des chaînes de réseaux, ordres héréditaires, suites de réseaux, β-extensions etc. avec des références précises. Soit P un sous-groupe parabolique composé des matrices triangulaires supérieures par blocs de type $(n, 1)$. On note M le sous-groupe de Levi de P constitué de matrices diagonales par blocs. Nous rappelons que $P(m)$ a le sens habituel. On note τ la représentation typique unique, apparaissant dans la représentation cuspidale σ. Nous pouvons ainsi supposer que χ est trivial.

Les représentations typiques se produisent comme des sous-représentations de la représentation

$$V_m := \mathrm{ind}_{P(m)}^{\mathrm{GL}_{n+1}(\mathcal{O}_F)}(\tau \boxtimes \mathrm{id}).$$

pour un entier $m \geq N$ (N sera défini explicitement plus tard). Pour des raisons qui seront expliquées plus tard, nous définissons un sous-groupe compact ouvert $P^0(m)$ de $P(m)$ tel que $P^0(m) \cap M = J^0 \times \mathcal{O}_F^\times$, $P^0(m) \cap U = P(m) \cap U$,

$P^0(m) \cap \bar{U} = P(m) \cap \bar{U}$ et $\lambda \boxtimes \mathrm{id}$ se prolonge à une représentation de $P^0(m)$. Nous montrons également que

$$\mathrm{ind}_{P^0(m)}^{\mathrm{GL}_{n+1}(\mathcal{O}_F)}(\lambda \boxtimes \mathrm{id}) \simeq \mathrm{ind}_{P(m)}^{\mathrm{GL}_{n+1}(\mathcal{O}_F)}(\tau \boxtimes \mathrm{id}).$$

Par récurrence sur m, $m \geq N$, nous montrons que les $\mathrm{GL}_{n+1}(\mathcal{O}_F)$ sous-représentations de V_m/V_1 sont non typiques.

Le type de Bushnell-Kutzko J_s est assez près de $P^0(N)$, dans le sens que $P^0(N) \cap P = J_s \cap P$ et de plus $\bar{U}(\varpi_F^N) \subset P^0(N) \cap \bar{U} \subset J_s \cap \bar{U} \subset \bar{U}(\varpi^{N-1}\,\mathcal{O}_F)$. En utilisant la décomposition de $\mathrm{ind}_{P^0(N)}^{J_s}(\mathrm{id})$, nous prouvons le théorème suivant.

Théorème 0.0.6. *Soit* Γ *une représentation typique de la classe inertielle* $s = [\mathrm{GL}_n(F) \times \mathrm{GL}_1(F), \sigma \boxtimes \chi]$ *et* $\#k_F > 2$*. La représentation* Γ *est unique et se plonge avec multiplicité un dans*

$$i_P^{\mathrm{GL}_{n+1}(F)}(\sigma \boxtimes \chi).$$

Résultats du chapitre 6

Dans ce chapitre, nous allons classifier les représentations typiques pour certaines classes inertielles

$$s = [\mathrm{GL}_2(F) \times \mathrm{GL}_2(F), \sigma_1 \boxtimes \sigma_2].$$

Soit P le sous-groupe parabolique composé de matrices triangulaires supérieures par blocs de type $(2,2)$. Soit M le sous-groupe de Levi de P et soit U le radical unipotent de P. On note $\bar{P}$ le sous-groupe parabolique opposé de P par rapport à M. Soit $\bar{U}$ le radical unipotent de $\bar{P}$. Soient (J_1^0, λ_1) et (J_2^0, λ_2) les types de Bushnell-Kutzko pour les classes inertielles $[\mathrm{GL}_2(F), \sigma_1]$ et $[\mathrm{GL}_2(F), \sigma_2]$ respectivement. Les groupes $P(m)$ auront le sens habituel pour $m \geq 1$. Nous définissons aussi un groupe $P^0(m)$ pour $m \geq N$ où N sera défini explicitement dans le texte principal du chapitre 6. Les groupes $P^0(m)$ ont leur décomposition d'Iwahori par rapport à P et M; $P^0(m) \cap \bar{U} = P(m) \cap \bar{U}$, $P^0(m) \cap U = P(m) \cap U$ et $P^0(m) \cap M = J_1^0 \times J_2^0$; et

$$\mathrm{ind}_{P(m)}^{\mathrm{GL}_4(\mathcal{O}_F)}(\tau_1 \boxtimes \tau_2) \simeq \mathrm{ind}_{P^0(m)}^{\mathrm{GL}_4(\mathcal{O}_F)}(\lambda_1 \boxtimes \lambda_2)$$

pour tous $m \geq N$.

Par des méthodes similaires à celles des chapitres 2, 3, 4, 5 nous allons réduire le problème de la classification des représentations typiques de s à trouver des représentations typiques se produisant en tant que sous-représentations de

$$\mathrm{ind}_{P^0(N+1)}^{\mathrm{GL}_4(\mathcal{O}_F)}(\lambda_1 \boxtimes \lambda_2). \tag{4}$$

Il se trouve que le type de Bushnell-Kutzko pour s n'est pas de la forme $(P^0(N+1), \lambda_1 \boxtimes \lambda_2)$ pour presque tous les cas. Cela signifie que nous ne pouvons pas conclure directement que les représentations typiques pour la classe inertielle s sont précisément les sous-représentations de (4).

Nous prévoyons (au moins lorsque $\#k_F > 2$) que les représentations typiques doivent se produire en tant que sous-représentations des

$$\mathrm{ind}_{J_s}^{\mathrm{GL}_4(\mathcal{O}_F)}(\lambda_1 \boxtimes \lambda_2).$$

Nous avons comparé les dimensions de la représentation ci-dessus avec celle de (4) et avons observé qu'elles sont en effet différentes et cela nous a donné la première heuristique pour s'attendre à ce qu'il y a des sous-représentations irréductibles non typiques qui se produisent dans (4). Nous avons essayé et même réussi dans de nombreux cas à classer les représentations typiques apparaissant dans (4). Nous allonons expliquer le résultat principal après avoir rappelé quelques aspects de la construction par Bushnell-Kutzko d'un type semi-simple pour s.

Soit $[\mathfrak{A}_1, n_1, 0, \beta_1]$ et $[\mathfrak{A}_2, n_2, 0, \beta_2]$ deux strates simples définissant les types de Bushnell-Kutzko (J_1^0, λ_1) et (J_2^0, λ_2) respectivement. On note e_1 et e_2 les indices de ramification des ordres héréditaires $\mathfrak{A}_1$ et $\mathfrak{A}_2$ respectivement. On note ϕ_i le facteur irréductible du polynôme caractéristique associé aux strates simples $[\mathfrak{A}_i, n_i, 0, \beta_i]$ pour $i \in \{1, 2\}$ (voir [BK93][Section 2.3]). Nous avons deux cas.

1. $n_1/e_1 \neq n_2/e_2$; $n_1/e_1 = n_2/e_2$ but $\phi_1 \neq \phi_2$.
2. $n_1/e_1 = n_2/e_2$ and $\phi_1 = \phi_2$

Les représentations σ_1 et σ_2 sont appelés **complètement distinctes** si elles satisfont à la condition (1). Sinon, elles sont dites avoir une **approximation commune**. Le cas d'approximation commune peut être divisé en 2 cas. Le premier est appelé cas homogène, approximation commune au niveau zéro. Le cas homogène dans notre situation actuelle (à la fois σ_1 et σ_2 sont des représentations de $\mathrm{GL}_2(F)$) signifie que $\mathfrak{A}_1 = \mathfrak{A}_2 := \mathfrak{A}$, $n_1 = n_2 := n$ et $\beta_1 = \beta_2 := \beta$. Et le caractère simple définissant l'extension β, κ, est isomorphe pour σ_1 and σ_2. Le deuxième cas est celui d'une approximation commune au niveau $l > 0$. En raison du manque de temps, nous ne traitons pas le cas où σ_1 et σ_2 ont approximation commune au niveau $l > 0$. Notre théorème principal est le suivant.

Théorème 0.0.7. *Soit $\#k_F > 3$ et s la classe inertielle*

$$[\mathrm{GL}_2(F) \times \mathrm{GL}_2(F), \sigma_1 \boxtimes \sigma_2]$$

où σ_1 et σ_2 sont complètement distinctes ou homogènes. Les représentations typiques pour la classe inertielle s sont précisément les sous-représentations *irréductibles de*

$$\mathrm{ind}_{J_s}^{\mathrm{GL}_4(\mathcal{O}_F)}(\lambda_s).$$

Nous allons d'abord examiner la classification des représentations typiques pour le cas homogène. Nous esquissons la preuve en détail.

Le type de Bushnell-Kutzko, (J_s, λ_s), pour la classe inertielle $s = [M, \sigma_1 \boxtimes \sigma_2]$, avec chaque σ_i contenant le type (J^0, λ_i) (qui est défini par les strates simples $[\mathfrak{A}, n, 0, \beta]$) est donnée par $\lambda_s = \lambda_1 \boxtimes \lambda_2$ et

$$J_s := \begin{pmatrix} J^0 & \mathcal{O}_E + \mathfrak{P}_{\mathfrak{A}}^{(n-t)} \\ \mathfrak{P}_E + \mathfrak{P}_{\mathfrak{A}}^{t+1} & J^0 \end{pmatrix}$$

pour $e = [n/2]$ et $E = F[\beta]$. Une observation importante est que $J_s \cap U \neq U(\mathcal{O}_F)$. Cela nous donne le problème principal. Pour attaquer cette situation, nous essayons d'abord de modifier les représentations induites (4) au niveau des sous-groupes à proximité de $P^0(N+1)$ et de J_s. Nous allons expliquer cela dans le cas non ramifié ($e(E|F) = 1$) où les notations sont déjà définies.

La première étape consiste à scinder la représentation

$$\rho_1 := \mathrm{ind}_{P^0(n+1)}^{P^0(t+1)}(\lambda_s).$$

L'idée est de remplacer le groupe $P^0(n+1)$ par le groupe J_s et de voir l'entrelacement entre ρ_1 et

$$\rho_2 := \mathrm{ind}_{J_s}^{P^0(t+1)}(\lambda_s).$$

Mais nous ne pouvons pas faire cela pour la raison très simple que $P^0(t)$ ne contient pas le groupe J_s. Alors nous utilisons un petit groupe J'_s tel que J'_s est contenu dans le groupe $P^0(t)$. Le seul changement entre J'_s et J_s est leur intersection avec le groupe unipotent inférieur. Ce groupe J'_s a la propriété que $P^0(n+1)J'_s = P^0(t)$ et la représentation

$$\rho_3 := \mathrm{ind}_{J'_s}^{P^0(t)}(\lambda_s)$$

est irréductible. Nous sommes en bonne situation pour la décomposition de Mackey, l'espace des opérateurs d'entrelacement entre ρ_1 et ρ_3 est de dimension un et tout opérateur d'entrelacement non nul est surjectif en raison de l'irréductibilité de ρ_3. Le reste de la preuve consiste à montrer que le noyau de cet opérateur d'entrelacement non trivial a des sous-représentations irréductibles qui apparaissent également dans ρ_1 pour certaines représentations λ_1 and λ_2.

L'opérateur d'entrelacement non trivial I entre ρ_1 et ρ_3 est donné par l'intégrale suivante:

$$I(f)(p) = \int_{u^- \in \ P(s,t) \cap \bar{U}} f(u^- p) du^-.$$

Si f est une fonction dans le noyau de I alors nous avons

$$\int_{u^-\in\ P^0(t)\cap\bar{U}} f(u^-u^+(u^-)^{-1}u^-)du^- = 0$$

pour tout $u^+ \in P^0(t)$. Soient u^- et u^+ représentés dans des matrices par blocs comme

$$u^- = \begin{pmatrix} \mathrm{id} & 0 \\ U^- & \mathrm{id} \end{pmatrix}, \quad u^+ = \begin{pmatrix} \mathrm{id} & U^+ \\ 0 & \mathrm{id} \end{pmatrix}$$

respectivement. L'équation intégrale peut être écrite comme

$$\int_{u^-\in\ P^0(t)\cap\bar{U}} \psi_{(\beta U^+ - U^+\beta)}(1+U^-)f(u^-)du^- = 0. \tag{5}$$

Nous notons tout d'abord que le groupe des caractères du groupe

$$P^0(t+1)/P^0(n+1) \simeq (P^0(t+1)\cap\bar{U})/(P^0(n+1)\cap\bar{U}) \simeq \mathfrak{P}_{\mathfrak{A}}^{t+1}/\mathfrak{P}_{\mathfrak{A}}^{n+1}$$

est isomorphe à $\mathfrak{P}_{\mathfrak{A}}^{-n}/\mathfrak{P}_{\mathfrak{A}}^{-t}$. Le noyau de I est engendré par les caractères qui ne sont pas dans l'image de $[\beta, \cdot]$. Nous allons utiliser le fait dans le lemme 6.2.3 pour montrer à l'aide des applications de co-restriction que les sous-représentations irréductibles de

$$\mathrm{ind}_{P^0(t)}^{\mathrm{GL}_4(\mathcal{O}_F)}(W),$$

où W est une sous-représentation de $\ker(I)$, sont non typiques.

Maintenant, il nous reste à comprendre les sous-représentations typiques de

$$\mathrm{ind}_{J_s'}^{\mathrm{GL}_4(\mathcal{O}_F)}(\lambda_s). \tag{6}$$

Puisque λ_s est une représentation de J_s en utilisant les techniques des chapitres précédents, nous pouvons montrer que la représentation

$$\mathrm{ind}_{J_s}^{\mathrm{GL}_4(\mathcal{O}_F)}(\lambda_s).$$

admet un complément Γ dans la représentation (6) tel que toutes les sous-représentations irréductibles de Γ ne sont pas typiques.

Maintenant, nous traitons le cas où σ_1 et σ_2 sont complètement distinctes. Une modification délicate de la représentation

$$\mathrm{ind}_{P^0(N+1)}^{\mathrm{GL}_n(\mathcal{O}_F)}(\lambda_1 \boxtimes \lambda_s)$$

donne la classification dans le cas où σ_1 et σ_2 sont complètement distinctes. Nous utilisons des techniques similaires à celles développées pour traiter le cas homogène. Ceci est la raison pour laquelle nous choisissons de mettre cette situation plutôt simple à la fin.

Chapter 1

Introduction

Let F be a non-discrete non-Archimedean locally compact field. This thesis is concerned with the study of the restriction of an irreducible smooth representation of $\mathrm{GL}_n(F)$ to a **maximal compact subgroup** K where $n \geq 2$. In particular we are interested in those irreducible representations (called **K-typical representations**) of K which determine the inertial support of the given irreducible smooth representation. In the context of local Langlands correspondence, such representations have seen significant arithmetic applications. In this thesis we try and achieve in many cases the classification of such irreducible smooth representations of K for a given inertial support s.

1.1 Motivation

Local Langlands Correspondence

Let $\bar{F}$ be a separable algebraic closure of F. Let F^{un} be the maximal unramified sub-extension of $\bar{F}$. We have the canonical quotient map

$$\mathrm{Gal}(\bar{F}/F) \to \mathrm{Gal}(F^{un}/F).$$

The group $\mathrm{Gal}(F^{un}/F)$ is canonically isomorphic to the Galois group of the residue field $k_{F^{un}}$ of F^{un} over k_F. Since $k_{F^{un}}$ is the algebraic closure of the finite field k_F we get the map

$$\mathrm{Gal}(\bar{F}/F) \to \mathrm{Gal}(k_{F^{un}}/k_F) \simeq \hat{\mathbb{Z}}. \tag{1.1}$$

Let q be the cardinality of the residue field k_F. We denote by Φ_F the automorphism of $k_{F^{un}}$ which sends an element x to x^q. Let W_F be the group consisting of those elements of $\mathrm{Gal}(\bar{F}/F)$ which map to a power of Φ_F under the map (1.1). The group W_F is called the Weil group of F. The group W_F can be made a locally compact group by declaring the open subgroups of $\mathrm{Gal}(\bar{F}/F^{un})$ (under its pro-finite topology) as open subgroups of W_F. Hence we obtain an exact sequence of topological groups

$$0 \to \mathrm{Gal}(\bar{F}/F^{un}) \to W_F \to \Phi_F^{\mathbb{Z}} \to 0$$

where $\Phi_F^{\mathbb{Z}}$ is given discrete topology.

Local class field theory gives us a canonical topological isomorphism

$$W_F^{ab} \simeq F^{\times}.$$

where W_F^{ab} is the quotient of W_F by the closure of the derived group of W_F. This gives us a one-to-one correspondence between the set of continuous characters of W_F and $F^\times$. The local Langlands correspondence establishes a higher dimensional analogue of the correspondence between the characters obtained via local class field theory. Such a correspondence can be stated with a certain algebraic object called Weil-Deligne representation. To begin with we introduce a norm $\|\|$ on the Weil group W_F. Let x be an element of W_F and the image of x under the map (1.1) be Φ_F^r then $\|x\| = q^{-r}$. An n-dimensional Weil Deligne representation is a triple (r, V, N) where V is an n-dimensional complex vector space, r is a homomorphism of W_F into $\mathrm{GL}(V)$ with open kernel, $N \in \mathrm{End}_{\mathbb{C}}(V)$ such that

$$r(x)Nr(x)^{-1} = \|x\|N$$

for all $x \in W_F$. We call the triple (r, V, N) Frobenius semi-simple if the representation (r, V) is semi-simple. The local Langlands conjecture (see [LRS93] for the case where characteristic of F is greater than zero and [HT01] and [Hen00] for the case where characteristic of F is zero) is a natural correspondence between the set of isomorphism classes of n-dimensional Frobenius semi-simple Weil-Deligne representations and the set of isomorphism classes of irreducible smooth complex representations of $\mathrm{GL}_n(F)$ (A representation (π, V) is called smooth if and only if the stabiliser of a vector $v \in V$ contains an open subgroup of $\mathrm{GL}_n(F)$ for the topology induced from F).

Let B_n be the set of pairs (M, σ) where M is a Levi-subgroup of a parabolic subgroup of $\mathrm{GL}_n(F)$ and σ is a supercuspidal representation of M. We recall that **inertial equivalence** is an equivalence relation on the set B_n defined by setting $(M_1, \sigma_1) \sim (M_2, \sigma_2)$ if and only if there exists an element $g \in G$ and an unramified character χ of M_2 such that $M_1 = gM_2g^{-1}$ and $\sigma_1^g \simeq \sigma_2 \otimes \chi$. We use the notation $[M, \sigma]$ for the equivalence class containing the pair (M, σ) the equivalence classes are also called as **inertial classes**. Every irreducible smooth representation π of $\mathrm{GL}_n(F)$ occurs as a sub-representation of a parabolically induced representation $i_P^{\mathrm{GL}_n(F)}(\sigma)$ where σ is a supercuspidal representation of a Levi-subgroup M of P. The pair (M, σ) is well determined upto $\mathrm{GL}_n(F)$-conjugacy (see [BZ77][Theorem 2.5 Theorem 2.9(a)(i)]). The class $[M, \sigma]$ is called the **inertial support** of π. (Inertial equivalence is defined for any reductive group G over F but we need it only for $\mathrm{GL}_n(F)$ in this thesis).

Given two triples (r_1, V_1, N_1) and (r_2, V_2, N_2), it turns out that the restrictions of r_1 and r_2 to the group $\mathrm{Gal}(\bar{F}/F^{un})$ are isomorphic if and only if the smooth representations π_1 and π_2 associated by the local Langlands correspondence to (r_1, V_1, N_1) and (r_2, V_2, N_2) respectively have the same inertial support. In several arithmetic applications (see [BM02] and [EG14] for instance) it is desirable to associate with a given inertial support say s an irreducible smooth representation τ of $\mathrm{GL}_n(\mathcal{O}_F)$ which has the property that

$\mathrm{Hom}_{\mathrm{GL}_n(\mathcal{O}_F)}(\tau, \pi) \neq 0$ implies that the inertial support of π is s and such a representation is called a $\mathrm{GL}_n(\mathcal{O}_F)$**-typical representation or typical representation**. We bring to the attention of the reader that we can expect at best an implication in one direction. One can produce easy examples for s such that there exists no irreducible smooth representation τ_s of $\mathrm{GL}_n(\mathcal{O}_F)$ such that $\mathrm{Hom}_{\mathrm{GL}_n(\mathcal{O}_F)}(\tau_s, \pi) \neq 0$ if and only if the inertial support of π is s.

Theory of Types

Let G be the group of F-rational points of an algebraic reductive group (examples being $\mathrm{GL}_n(F)$, $\mathrm{SL}_n(F)$ and $\mathrm{SO}(V, q)$ for some finite dimensional quadratic space (V, q) over F, etc). It was shown by Bernstein that the category of smooth representations $\mathcal{M}(G)$ admits a decomposition

$$\mathcal{M}(G) = \prod_{s \in \mathcal{B}(G)} \mathcal{M}_s(G)$$

where $\mathcal{M}_s(G)$ is the full sub-category consisting of smooth representations with all their irreducible sub-quotients having inertial support s. The theory of types developed initially by Bushnell-Kutzko (see [BK98] for a general discussion on smooth representations via types) gives a construction of pairs (J_s, λ_s) where J_s is a compact open subgroup of G and λ_s is a smooth irreducible representation of J_s such that $\mathrm{Hom}_{J_s}(\lambda_s, \pi) \neq 0$ if and only if $\pi \in \mathcal{M}_s(G)$ for all irreducible smooth representations π of G and such a pair (J_s, λ_s) is called a **type** for s. Such a type (J_s, λ_s) gives a natural equivalence of categories $\mathcal{M}_s(G)$ and the category of modules over the spherical Hecke algebra $\mathcal{H}(J_s, \lambda_s)$.

Let K be a maximal compact subgroup of G and s be an inertial class of G. If we know the existence of a type (J_s, λ_s) such that $J_s \subset K$ then by Frobenius reciprocity any irreducible sub-representation of

$$\mathrm{ind}_{J_s}^{K}(\lambda_s) \tag{1.2}$$

if contained in a smooth irreducible representation π of G then π contains the representation λ_s on restriction to the group J_s and hence the inertial support of π is s. This shows that irreducible sub-representations of (1.2) are K-typical representations. As types are not known to exist in every case the following natural questions appear :

1. Does there exist a K-typical representation?
2. For a given inertial class s is the cardinality of K-typical representations finite?
3. What are all typical representations?

For $G = \mathrm{GL}_n(F)$ types (J_s, λ_s) are explicitly constructed by Bushnell-Kutzko in the articles [BK93] and [BK99]. For $\mathrm{GL}_n(F)$ we may and do choose "the Bushnell-Kutzko type" (J_s, λ_s) such that $J_s \subset \mathrm{GL}_n(\mathcal{O}_F)$ for all $s \in \mathcal{B}_n$. In this thesis we will see that the above questions for $G = \mathrm{GL}_n(F)$ can be answered in terms of and by the use of theory of types. For $\#k_F > 3$ we show in many cases that the irreducible sub-representations of

$$\mathrm{ind}_{J_s}^{\mathrm{GL}_n(\mathcal{O}_F)}(\lambda_s)$$

are precisely the typical representations for the component s. In this sense we classify typical representations for the component s. We bring to attention of the reader that the types constructed by Bushnell-Kutzko may not be unique even up to conjugation. We use the terminology "the Bushnell-Kutzko type" for the pair (J_s, λ_s) a type for $s = [M, \sigma]$ constructed by the inductive procedure in the article [BK99] after fixing a type for the inertial class $[M, \sigma]$ of M.

There can be various constructions of types (J, λ) for a given component s (in the sense that the pair (J, λ) has the property $\mathrm{Hom}_J(\lambda, \pi) \neq 0$ if and only if the inertial support of π is s for any irreducible smooth representation π of G). For any such construction and K a maximal contact subgroup containing J, the irreducible sub-representations of

$$\mathrm{ind}_J^K(\lambda)$$

are a K-typical representation. Hence the theory of typical representation, at least for the case of GL_n, aims to give a uniform approach. It could be interesting to prove at least the finiteness of typical representations in general case.

1.2 Known results

The case of $\mathrm{GL}_2(F)$ is treated by Henniart in the appendix to the article [BM02]. He completely classified typical representations for all possible inertial classes. Henniart predicted that his results can be extended to $\mathrm{GL}_n(F)$ by similar techniques he used at least in those cases where the underlying Levi-subgroup of the inertial class s is $\mathrm{GL}_n(F)$. Paskunas has classified the typical representations for the inertial classes $[\mathrm{GL}_n(F), \sigma]$. We now describe the results of Henniart and Paskunas. Before going any further we note that J_s can be conjugated to a subgroup of $\mathrm{GL}_n(\mathcal{O}_F)$ and we assume that indeed J_s is a subgroup of $\mathrm{GL}_n(\mathcal{O}_F)$.

Theorem 1.2.1 (Henniart). *Let s be an inertial class for $GL_2(F)$. Let (J_s, λ_s) be the Bushnell-Kutzko type for the inertial class s. If $\#k_F > 2$ or $s = [GL_2(F), \sigma]$ then the typical representations for s occur as sub-representations* *of*

$$\mathrm{ind}_{J_s}^{\mathrm{GL}_2(\mathcal{O}_F)}(\lambda_s).$$

Let T be the maximal torus of $\mathrm{GL}_2(F)$ consisting of diagonal matrices and $s = [T, \chi]$ be an inertial class for $\mathrm{GL}_2(F)$. Let us identify T with $F^\times \times F^\times$ and the character χ be $\chi((a, b)) = \chi_1(a)\chi_2(b)$ for two characters χ_1 and χ_2 of $F^\times$. Let B be a Borel subgroup containing T. Let $B(m)$ be the group of matrices of $\mathrm{GL}_2(\mathcal{O}_F)$ which under the mod $\mathfrak{P}_F^m$ reduction lie in the group $B(\mathcal{O}_F/\mathfrak{P}_F^m)$. If $\chi_1\chi_2^{-1} \neq \mathrm{id}$, N is the level of $\chi_1\chi_2^{-1}$ and $\#k_F = 2$ then Henniart showed that

$$\mathrm{ind}_{B(N)}^{\mathrm{GL}_2(\mathcal{O}_F)}(\chi)$$

and

$$\mathrm{ind}_{B(N+1)}^{\mathrm{GL}_2(\mathcal{O}_F)}(\chi)$$

are typical for the inertial class s. The Bushnell-Kutzko type for the inertial class $s = [T, \chi]$ is given by $(B(N), \chi)$. So this shows that in the present case there are indeed additional representations other than those irreducible sub-representations of (1.2) which are typical for s. For all other inertial classes $[T, \chi]$ the typical representations are shown to be sub-representations of (1.2).

For the inertial class $s = [\mathrm{GL}_n(F), \sigma]$ it follows easily that

$$\mathrm{ind}_{J_s}^{\mathrm{GL}_n(\mathcal{O}_F)}(\lambda_s)$$

is an irreducible representation and Paskunas in the article [Pas05] showed that

Theorem 1.2.2 (Paskunas)**.** *For any positive integer $n > 1$ and for any inertial class $s = [\mathrm{GL}_n(F), \sigma]$ there exists a unique typical representation.*

1.3 Results of this thesis

In this thesis we are interested in the classification of typical representations for the inertial classes $[M, \sigma]$ where M is a proper Levi-subgroup of $\mathrm{GL}_n(F)$ and $n \geq 3$. Here we assume the Bushnell- Kutzko type (J_s, λ_s) is conjugated such that $J_s \subset \mathrm{GL}_n(\mathcal{O}_F)$. We give a detailed description of the results from each chapter. Our goal is to describe the results and method of proof briefly.

Results of Chapter 2

If τ is a typical representation for an inertial class s then we show that τ occurs in a smooth irreducible representation say $\pi \in \mathcal{M}_s(G)$ of $\mathrm{GL}_n(F)$. We choose a representative (M, σ) for s such that M is the Levi-subgroup consisting of block diagonal matrices and σ naturally a supercuspidal representation of M. Now the representation π occurs in some representation

$$i_P^{\mathrm{GL}_n(F)}(\sigma \otimes \chi)$$

where P is any parabolic subgroup containing M as a Levi-subgroup and $i_P^{\mathrm{GL}_n(F)}$ denotes the parabolic induction and χ is an unramified character of M. Hence for the classification of typical representations we have to look for the typical representations occurring in the representation

$$\mathrm{res}_{\mathrm{GL}_n(\mathcal{O}_F)}\{i_P^{\mathrm{GL}_n(F)}(\sigma)\} \simeq \mathrm{ind}_{P \cap \mathrm{GL}_n(\mathcal{O}_F)}^{\mathrm{GL}_n(\mathcal{O}_F)}(\sigma). \tag{1.3}$$

Now we identify M with the product $\mathrm{GL}_{n_1}(F) \times \mathrm{GL}_{n_2}(F) \times \ldots \mathrm{GL}_{n_r}(F)$ for some ordered partition $(n_1, n_2, \ldots, n_r)$ of n and σ as a tensor product $\sigma_1 \boxtimes \sigma_2 \boxtimes \cdots \boxtimes \sigma_r$ where σ_i is a cuspidal representation of $\mathrm{GL}_{n_i}(F)$. Let τ_i be the unique typical representation occurring in σ_i. It was observed by Will Conley that the representation

$$\mathrm{ind}_{P \cap \mathrm{GL}_n(\mathcal{O}_F)}^{\mathrm{GL}_n(\mathcal{O}_F)}(\boxtimes_{i=1}^r \tau_i)$$

admits a complement in (1.3) whose $\mathrm{GL}_n(\mathcal{O}_F)$-irreducible sub-representations are non-typical for the component s. We extend Conley's result, to be used for proofs by induction on n. But for sake of brevity we cannot go into details here.

Now any typical representation occurs among the $\mathrm{GL}_n(\mathcal{O}_F)$-irreducible sub-representations of

$$\mathrm{ind}_{P \cap \mathrm{GL}_n(\mathcal{O}_F)}^{\mathrm{GL}_n(\mathcal{O}_F)}(\boxtimes_{i=1}^r \tau_i).$$

This representation is still an infinite dimensional representation. The first idea is to construct compact open subgroups groups H_m for $m \geq 1$ such that $H_{m+1} \subset H_m$, $\cap_{m \geq 1} H_m = P \cap \mathrm{GL}_n(\mathcal{O}_F)$ and $\boxtimes_{i=1}^r \tau_i$ extends to a representation of H_1. With some additional conditions we show that

$$\mathrm{ind}_{P \cap \mathrm{GL}_n(\mathcal{O}_F)}^{\mathrm{GL}_n(\mathcal{O}_F)}(\boxtimes_{i=1}^r \tau_i) \simeq \bigcup_{m \geq 1} \mathrm{ind}_{H_m}^{\mathrm{GL}_n(\mathcal{O}_F)}(\boxtimes_{i=1}^r \tau_i).$$

Now depending on the inertial support we have to define H_m in each chapter and analyse the representations

$$\mathrm{ind}_{H_m}^{\mathrm{GL}_n(\mathcal{O}_F)}(\boxtimes_{i=1}^r \tau_i).$$

In addition to this we show several technical lemmas which are frequently used in the entire thesis.

Results of Chapter 3

This chapter concerns those components (called **level zero inertial classes**)

$$[M = \prod_{i=1}^{r} \mathrm{GL}_{n_i}(F), \boxtimes_{i=1}^{r}\sigma_i]$$

where each σ_i contains a non-zero vector fixed by the principal congruence subgroup of level one. In other words the Bushnell-Kutzko type for σ_i is given by $(\mathrm{GL}_{n_i}(\mathcal{O}_F), \tau_i)$ where τ_i is obtained by inflating a cuspidal representation of $\mathrm{GL}_{n_i}(k_F)$ for all $i \leq r$. Now choose P to be the group of block upper diagonal matrices containing M as a Levi-subgroup. For a positive integer m we denote by $P(m)$ the group of matrices in $\mathrm{GL}_n(\mathcal{O}_F)$ which under the mod-$\mathfrak{P}_F^m$ reduction lie inside the group $P(\mathcal{O}_F/\mathfrak{P}_F^m)$. Now the representation $\boxtimes_{i=1}^{r}\tau_i$ extends to a representation of $P(1)$ by inflation. The sequence of groups H_m we described in the earlier sub-section are given by $P(m)$.

To simplify the notation we denote by V_m the representation

$$\mathrm{ind}_{P(m)}^{\mathrm{GL}_n(\mathcal{O}_F)}(\boxtimes_{i=1}^{r}\tau_i).$$

Using induction on the positive integer m we show the theorem

Theorem 1.3.1. *The $\mathrm{GL}_n(\mathcal{O}_F)$-irreducible sub-representations of V_m/V_1 are not typical for the inertial class $s = [M, \sigma]$.*

We actually construct a complement of V_1 in V_m.

We note that the Bushnell-Kutzko type for the inertial class s is given by the pair $(P(1), \boxtimes_{i=1}^{r}\tau_i)$, from which we can conclude that the above theorem completely classifies the typical representations in this case. Through our analysis we gain some additional information. We conclude the result in the following theorem.

Theorem 1.3.2. *Let $s = [M, \sigma]$ be a level zero inertial class. Let Γ be a typical representation for the inertial class s. The representation Γ is an irreducible sub-representation of V_1 and*

$$\dim_{\mathbb{C}} \mathrm{Hom}_{\mathrm{GL}_n(\mathcal{O}_F)}(\Gamma, V_1) = \dim_{\mathbb{C}} \mathrm{Hom}_{\mathrm{GL}_n(\mathcal{O}_F)}(\Gamma, i_P^{\mathrm{GL}_n(F)}(\sigma)).$$

We now sketch the proof of the above theorems. The essential features of the proof are captured in two cases, the first case: M is isomorphic to $\mathrm{GL}_{n_1}(F) \times \mathrm{GL}_{n_2}(F)$ and the the second case is: M is the diagonal torus of $\mathrm{GL}_n(F)$.

To begin with lets consider the case $n_1 = n_2 = 1$. In this case Henniart in the article [BM02][Appendix] uses Casselman's description (see [Cas73]) of the complete decomposition of the restriction of a smooth representation to the maximal compact subgroup $\mathrm{GL}_2(\mathcal{O}_F)$. Casselman shows that two smooth representations with the same central character have isomorphic restriction to $\mathrm{GL}_2(\mathcal{O}_F)$ except for a finite part. He effectively controls this finite part as well. It turns out in most of the cases (at least in the principal series case) that this finite part is

$$\mathrm{ind}_{J_s}^{\mathrm{GL}_2(\mathcal{O}_F)}(\lambda_s).$$

The rest of the restriction depends only on the conductor and the central character. Henniart manages to produce two smooth representations with same central character and conductor but changing the inertial support. This gives the result. Although additional work has to be carried out when $\#k_F = 2$.

Let $n_1 + n_2 > 2$. When $n \geq 3$ we do not in general know the complete decomposition into irreducible summands of the restriction of an irreducible smooth representations to the maximal compact subgroup $\mathrm{GL}_n(\mathcal{O}_F)$. In practice we found that such result is not required for our purpose. We already reduced to check for typical representations occurring in the representations

$$\mathrm{ind}_{P(m)}^{\mathrm{GL}_n(\mathcal{O}_F)}(\tau_1 \boxtimes \tau_2)$$

for $m \geq 1$. Notice that we have

$$\mathrm{ind}_{P(m+1)}^{\mathrm{GL}_n(\mathcal{O}_F)}(\tau_1 \boxtimes \tau_2) \simeq \mathrm{ind}_{P(m)}^{\mathrm{GL}_n(\mathcal{O}_F)}\{\mathrm{ind}_{P(m+1)}^{P(m)}(\mathrm{id}) \otimes (\tau_1 \boxtimes \tau_2)\}. \tag{1.4}$$

By Frobenius reciprocity we know that id occurs with multiplicity one in the representation

$$\mathrm{ind}_{P(m+1)}^{P(m)}(\mathrm{id}).$$

By means of Clifford theory we achieve the following decomposition,

$$\mathrm{ind}_{P(m+1)}^{P(m)}(\mathrm{id}) \simeq \mathrm{id} \oplus \bigoplus_{j=1}^{t} \mathrm{ind}_{Z_j}^{P(m)}(U_j)$$

where Z_j is a compact open subgroup which is “small enough”. We will come back to what we mean by “small enough". Now let us return to 1.4. We have

$$\mathrm{ind}_{P(m+1)}^{\mathrm{GL}_n(\mathcal{O}_F)}(\tau_1 \boxtimes \tau_2) \simeq \mathrm{ind}_{P(m)}^{\mathrm{GL}_n(\mathcal{O}_F)}(\tau_1 \boxtimes \tau_2) \oplus \bigoplus_{1 \leq j \leq t} \mathrm{ind}_{Z_j}^{\mathrm{GL}_n(\mathcal{O}_F)}\{U_j \otimes (\tau_1 \boxtimes \tau_2)\}.$$

For some fixed j we wish to compare the terms $\mathrm{ind}_{Z_j}^{\mathrm{GL}_n(\mathcal{O}_F)}\{U_j \otimes (\tau_1 \boxtimes \tau_2)\}$ for various τ_1 and τ_2. It is exactly in this context that we refer to Z_j being “small enough". We have shown that for every irreducible sub-representation ξ of $\mathrm{res}_{Z_j}(\tau_1 \boxtimes \tau_2)$ we can find an irreducible representation $\tau_1' \boxtimes \tau_2'$ such that

1. $\tau_1' \boxtimes \tau_2'$ is the inflation of a non-cuspidal representation of $M(k_F)$.

2. ξ occurs in the representation $\mathrm{res}_{Z_j}(\tau_1' \boxtimes \tau_2')$.

This is enough to show that the irreducible sub-representations of

$$\mathrm{ind}_{Z_j}^{\mathrm{GL}_n(\mathcal{O}_F)}\{U_j \otimes (\tau_1 \boxtimes \tau_2)\}$$

are non-typical. By means of induction on the positive integer m we prove the theorem.

The above attempt does not work if both τ_1 and τ_2 are characters. The reason is that $M(k_F)$ has no non-cuspidal representations in this case. For the case of principal series case i.e. $s = [T, \chi]$ (T is the maximal torus consisting of invertible diagonal matrices of dimension $n > 2$) we use slightly different techniques. We assume that we know the result for $n-1$ and show that typical representations occur only in

$$W_m := \mathrm{ind}_{R(m)}^{\mathrm{GL}_n(\mathcal{O}_F)}\{(\mathrm{ind}_{B_{n-1}(1)}^{\mathrm{GL}_{n-1}(\mathcal{O}_F)}(\boxtimes_{i=1}^{n-1}(\chi_i))) \boxtimes \chi_n\}$$

where R is a parabolic subgroup of the type $(n-1, 1)$ and B_{n-1} is the Borel subgroup consisting of upper triangular matrices. We now use induction on m as done earlier to show that the typical representations occur as sub-representations of W_1. The induction step is achieved by comparing the terms

$$\mathrm{ind}_{Z_j}^{\mathrm{GL}_n(\mathcal{O}_F)}\{U_j \otimes (\mathrm{ind}_{B_{n-1}(1)}^{\mathrm{GL}_{n-1}(\mathcal{O}_F)}(\boxtimes_{i=1}^{n-1}(\chi_i)) \boxtimes \chi_n)\}. \tag{1.5}$$

The mod $\mathfrak{P}_F$ reduction of the group $Z_j \cap N$ (N is the Levi-subgroup of R consisting of block diagonal matrices of size $(n-1, 1)$) is contained in the following subgroup

$$\left\{ \begin{pmatrix} A & B & 0 \\ 0 & d & 0 \\ 0 & 0 & d \end{pmatrix} | A \in \mathrm{GL}_{n-2}(k_F); B \in M_{n-2\times 1}(k_F); d \in k_F^\times \right\}.$$

We will further decompose the representation 1.5 by first decomposing the restriction of the representation

$$\mathrm{ind}_{B_{n-1}(1)}^{\mathrm{GL}_{n-1}(\mathcal{O}_F)}(\boxtimes_{i=1}^{n-1}(\chi_i))$$

to the group

$$P_{(n-2,1)}(k_F) := \left\{ \begin{pmatrix} A & B \\ 0 & d \end{pmatrix} | A \in \mathrm{GL}_{n-2}(k_F); B \in M_{n-2\times 1}(k_F); d \in k_F^\times \right\}.$$

We then compare the terms

$$\mathrm{ind}_{Z_j}^{\mathrm{GL}_n(\mathcal{O}_F)}\{U_j \otimes (\gamma_p \boxtimes \chi_n)\}$$

where γ_p is the inflation of an irreducible representation of

$$\mathrm{res}_{P_{(n-2,1)}(k_F)}\{\mathrm{ind}_{B_{n-1}(k_F)}^{\mathrm{GL}_{n-1}(k_F)}(\boxtimes_{i=1}^{n-1}\chi_i)\}.$$

We combine these ideas to complete the proof of theorem 3.0.9.

Results of Chapter 4

Let T_n be the maximal torus, consisting of invertible diagonal matrices in $\mathrm{GL}_n(F)$. For $\#k_F > 3$ we classify the typical representations for the components $s = [T_n, \chi]$. We will show that the typical representations occur as sub-representations of

$$\mathrm{ind}_{J_s}^{\mathrm{GL}_n(\mathcal{O}_F)}(\lambda_s).$$

First our goal is to define the groups H_m such that $H_{m+1} \subset H_m$ and

$$\cap_{m\geq 1} H_m = B_n(\mathcal{O}_F)$$

where B_n is the Borel subgroup consisting of invertible upper triangular matrices. We note that the Bushnell-Kutzko type (J_s, λ_s) in this case has the property that $J_s \cap B_n = B_n(\mathcal{O}_F)$. We can hence choose $H_1 = J_s$ and the other groups need to be defined more carefully. We sketch the details.

We identify T_n with the group $\times_{i=1}^{n} F^{\times}$ and the character χ with $\boxtimes_{i=1}^{n}\chi_i$ where χ_i is a character of $F^{\times}$. We denote by $l(\chi)$ the least positive integer k such that $1+\mathfrak{P}_F^k$ is contained in the kernel of χ. Let $J_\chi(m)$ be the set consisting of matrices (a_{ij}) where $a_{ij} \in \mathfrak{P}_F^{l(\chi_i\chi_j^{-1})+m-1}$ for all $i > j$, $a_{ii} \in \mathcal{O}_F^{\times}$ and $a_{ij} \in \mathcal{O}_F$ for all $i < j$. We will show that $J_\chi(m)$ is indeed a compact open subgroup and we also establish the following

1. $J_s = J_\chi(1)$
2. $\cap_{m\geq 1} J_\chi(m) = B_n(\mathcal{O}_F)$
3. The character χ of $T(\mathcal{O}_F)$ extends to a character of $J_\chi(1)$.

Hence we define $H_m = J_\chi(m)$ for $m \geq 1$. This shows that

$$\mathrm{res}_{\mathrm{GL}_n(\mathcal{O}_F)}\, i_{B_n}^{\mathrm{GL}_n(F)}(\chi) \simeq \bigcup_{m\geq 1} \mathrm{ind}_{J_\chi(m)}^{\mathrm{GL}_n(\mathcal{O}_F)}(\chi)$$

We denote by $V_m(\chi)$ the representation

$$\mathrm{ind}_{J_\chi(m)}^{\mathrm{GL}_n(\mathcal{O}_F)}(\chi).$$

By using induction on the positive integers n and m we show the following theorem:

Theorem 1.3.3. *The $\mathrm{GL}_n(\mathcal{O}_F)$-irreducible sub-representations of $V_m(\chi)/V_1(\chi)$ are not typical for the component $s = [T_n, \chi]$.*

From the induction hypothesis on the positive n we can show that the typical representations occur in the following sub-representation of V_m:

$$\mathrm{ind}_{J_\chi(1,m)}^{\mathrm{GL}_n(\mathcal{O}_F)}(\chi).$$

The group $J_\chi(1,m)$ contains $J_\chi(m)$ and $\cap_{\geq m} J_\chi(1,m)$ is of the form

$$\left\{\begin{pmatrix} A & B \\ 0 & c \end{pmatrix} | A \in J_{\boxtimes_{i=1}^{n-1}\chi_i}(1); B \in M_{n-1\times 1}(\mathcal{O}_F); c \in \mathcal{O}_F^\times\right\}.$$

Now as seen earlier we decompose the representation

$$\mathrm{ind}_{J_\chi(1,m+1)}^{J_\chi(1,m)}(\mathrm{id})$$

as follows

$$\mathrm{ind}_{J_\chi(1,m+1)}^{J_\chi(1,m)}(\mathrm{id}) \simeq \mathrm{id} \oplus \bigoplus_{1\leq j\leq p} \mathrm{ind}_{Z_j}^{J_\chi(1,m)}(U_j)$$

and show that $\mathrm{res}_{Z_j\cap T_n}(\chi) = \mathrm{res}_{Z_j\cap T_n}(\chi')$ for some character χ' such that $[T,\chi'] \neq [T,\chi]$ and $J_\chi(1) = J_{\chi'}(1)$. This shows that the irreducible sub-representations of

$$\mathrm{ind}_{Z_j}^{\mathrm{GL}_n(\mathcal{O}_F)}(U_j \otimes \chi)$$

are not typical. This proves the main result by induction on m. We bring to the attention of the reader that the decomposition of the group

$$\mathrm{ind}_{J_\chi(m+1)}^{J_\chi(m)}(\mathrm{id})$$

is much more involved than the decomposition of the corresponding representation obtained by replacing $J_\chi(m)$ and $J_\chi(m+1)$ by $J_\chi(1,m)$ and $J_\chi(1,m+1)$ respectively. This is the reason why we adapt to do induction on both the variables n and m.

Review of chapter 5

In this chapter we are interested in classification of typical representations for the inertial classes $s = [\mathrm{GL}_n(F) \times GL_1(F), \sigma \boxtimes \chi]$. In this chapter and the next we will use the apparatus of [BK93] like lattice chains, hereditary orders, stata, simple characters, β-extensions etc with precise references. Let P be a parabolic subgroup consisting of block upper triangular matrices of the type $(n,1)$. We denote by M the Levi-subgroup of P consisting of block diagonal matrices. We recall that $P(m)$ has the usual meaning. We denote by τ the unique typical representation occurring in the cuspidal representation σ. We can as well assume that χ is trivial.

Typical representations occur as sub-representations of the representation

$$V_m := \mathrm{ind}_{P(m)}^{\mathrm{GL}_{n+1}(\mathcal{O}_F)}(\tau \boxtimes \mathrm{id}).$$

for some positive integer $m \geq N$ (N will be explicitly defined later). Let $[\mathfrak{A}, l, 0, \beta]$ be a simple stratum defining the Bushnell-Kutzko type (J^0,λ) for the component $[\mathrm{GL}_n(F),\sigma]$. The representation τ is isomorphic to

$$\mathrm{ind}_{J^0}^{\mathrm{GL}_{n+1}(\mathcal{O}_F)}(\lambda).$$

As we did in the previous situations we will need the decomposition of the representation
$$\mathrm{ind}_{P(m+1)}^{P(m)}(\mathrm{id}).$$
The above representation is decomposed in chapter 3 as
$$\mathrm{ind}_{P(m+1)}^{P(m)}(\mathrm{id}) \simeq \mathrm{id} \oplus \bigoplus_j \mathrm{ind}_{Z'_j}^{P^0(m)}(U_j).$$

Now recall that the groups $Z'_j \cap M$ might be smaller than $M(\mathcal{O}_F)$. Now $\mathrm{res}_{Z'_j \cap M}(\tau \boxtimes \mathrm{id})$ would involve complicated Mackey decompositions and moreover the structure of $(J^0 \times {\mathcal{O}_F}^{\times})^u \cap (J^0 \times {\mathcal{O}_F}^{\times})$ can be complicated for a given $u \in \mathrm{GL}_n(\mathcal{O}_F) \times {\mathcal{O}_F}^{\times}$. To overcome this problem we define a compact open subgroup $P^0(m)$ of $P(m)$ such that $P^0(m) \cap M = J^0 \times \mathcal{O}_F^{\times}$, $P^0(m) \cap U = P(m) \cap U$, $P^0(m) \cap \bar{U} = P(m) \cap \bar{U}$ and $\lambda \boxtimes \mathrm{id}$ extends to a representation of $P^0(m)$. We also show that
$$\mathrm{ind}_{P^0(m)}^{\mathrm{GL}_{n+1}(\mathcal{O}_F)}(\lambda \boxtimes \mathrm{id}) \simeq \mathrm{ind}_{P(m)}^{\mathrm{GL}_{n+1}(\mathcal{O}_F)}(\tau \boxtimes \mathrm{id}).$$
Now we show that
$$\mathrm{ind}_{P^0(m)}^{P^0(m+1)}(\mathrm{id}) \simeq \mathrm{id} \oplus \bigoplus_{1 \leq j \leq p} \mathrm{ind}_{Z_j}^{P^0(m)}(U_j)$$
where $Z_j \cap M$ contained a priori in $J^0 \times \mathcal{O}_F^{\times}$ is "small enough" in this group.

To explain the meaning of "small enough" we need to recall certain aspects of the type (J^0, λ). Let B be the algebra $\mathrm{End}_{F[\beta]}(\mathfrak{A} \otimes F)$. Let $\mathfrak{B}$ be the order $\mathrm{End}_{F[\beta]}(\mathfrak{A} \otimes F) \cap \mathfrak{A}$. The group J^0 has a normal subgroup J^1 such that $J^0/J^1 \simeq U^0(\mathfrak{B})/U^1(\mathfrak{B})$. The group $U^0(\mathfrak{B})/U^1(\mathfrak{B}) \simeq \mathrm{GL}_{n'}(k_{F[\beta]})$. The representation λ is isomorphic to $\kappa \otimes \lambda'$ where λ' is a cuspidal representation of J^0/J^1 and κ is a certain representation called β-extension. We refer to $Z_j \cap M$ as "small enough" in the sense that $Z_j \cap (U^0(\mathfrak{B}) \times \mathcal{O}_F^{\times})$ satisfies the important property: when $n' > 1$ for every irreducible sub-representation ξ of
$$\mathrm{res}_{Z_j \cap (U^0(\mathfrak{B}) \times \mathcal{O}_F^{\times})}(\kappa \otimes \lambda')$$
there exists a non-cuspidal representation λ'' of J^0/J^1 such that ξ occurs in
$$\mathrm{res}_{Z_j \cap (U^0(\mathfrak{B}) \times \mathcal{O}_F^{\times})}(\kappa \otimes \lambda'').$$
With this we conclude that any irreducible sub-representation Γ of
$$\mathrm{ind}_{Z_j}^{\mathrm{GL}_{n+1}(\mathcal{O}_F)}(\lambda \boxtimes \mathrm{id}) \tag{1.6}$$
occurs as an irreducible representations of
$$\mathrm{ind}_{Z_j}^{\mathrm{GL}_{n+1}(\mathcal{O}_F)}(\tau' \boxtimes \mathrm{id})$$

(τ' may depend on Γ) where we show that irreducible sub-representations of τ' occur in the restriction to $\mathrm{GL}_n(\mathcal{O}_F)$ of an irreducible non-cuspidal representation of $\mathrm{GL}_n(F)$. For showing that τ' occurs in the restriction to $\mathrm{GL}_{n+1}(\mathcal{O}_F)$ of an irreducible non-cuspidal representation we use the novel feature of simple characters and their compatibility with change of rings ($\mathfrak{A}$) due to Bushnell-Kutzko (see [BK93][Section 3.6, Proposition 8.3.5]). This shows that irreducible sub-representations of 1.6 are not typical representations. Hence by using induction on integer m, $m \geq N$, we can show that $\mathrm{GL}_{n+1}(\mathcal{O}_F)$ sub-representations of V_m/V_1 are non-typical.

The Bushnell-Kutzko type J_s is almost close enough to $P^0(N)$. In the sense that $P^0(N) \cap P = J_s \cap P$ and moreover

$$\bar{U}(\varpi_F^N) \subset P^0(N) \cap \bar{U} \subset J_s \cap \bar{U} \subset \bar{U}(\varpi^{N-1}\,\mathcal{O}_F).$$

We decompose the representation

$$\mathrm{ind}_{P^0(N)}^{J_s}(\mathrm{id}) = \mathrm{id} \oplus \bigoplus_j \mathrm{ind}_{Z_j}^{J_s}(U_j)$$

for "small enough" groups Z_j and show that irreducible sub-representations of

$$\mathrm{ind}_{Z_j}^{\mathrm{GL}_{n+1}(\mathcal{O}_F)}\{U_j \otimes (\lambda \boxtimes \mathrm{id})\}$$

are not typical representations. This gives the result:

Theorem 1.3.4. *Let Γ be a typical representation for the inertial class*

$$s = [\mathrm{GL}_n(F) \times \mathrm{GL}_1(F), \sigma \boxtimes \chi]$$

and $\#k_F > 2$. The representation Γ is unique and occurs with a multiplicity one in the representation

$$i_P^{\mathrm{GL}_{n+1}(F)}(\sigma \boxtimes \chi).$$

We show that typical representations should occur as sub-representations of

$$\mathrm{ind}_{J_s}^{\mathrm{GL}_{n+1}(\mathcal{O}_F)}(\lambda \boxtimes \chi).$$

We know from the Bushnell-Kutzko theory that the above representation is irreducible. This gives our uniqueness of typical representation. The multiplicity one result is not known to the author without the results of this thesis. We note that typical representations for the inertial classes of $\mathrm{GL}_3(F)$ where $\#k_F > 3$ are precisely the sub-representations of

$$\mathrm{ind}_{J_s}^{\mathrm{GL}_3(\mathcal{O}_F)}(\lambda_s).$$

Review of chapter 6

In this chapter we will classify typical representations for certain inertial classes

$$s = [\mathrm{GL}_2(F) \times \mathrm{GL}_2(F), \sigma_1 \boxtimes \sigma_2].$$

Let P be the parabolic subgroup consisting of block upper triangular matrices of the type $(2,2)$. Let M be the Levi-subgroup of P and U be the unipotent radical of P. We denote by $\bar{P}$ the opposite parabolic subgroup of P with respect to M. Let $\bar{U}$ be the unipotent radical of $\bar{P}$. Let (J_1^0, λ_1) and (J_2^0, λ_2) be Bushnell-Kutzko's types for the inertial classes $[\mathrm{GL}_2(F), \sigma_1]$ and $[\mathrm{GL}_2(F), \sigma_2]$ respectively. The groups $P(m)$ will have the usual meaning for $m \geq 1$. We also define a group $P^0(m)$ for $m \geq N$ where N will be defined explicitly in the main text of the chapter 6. The groups $P^0(m)$ has Iwahori decomposition with respect to P and M; $P^0(m) \cap \bar{U} = P(m) \cap \bar{U}$, $P^0(m) \cap U = P(m) \cap U$ and $P^0(m) \cap M = J_1^0 \times J_2^0$; and

$$\mathrm{ind}_{P(m)}^{\mathrm{GL}_4(\mathcal{O}_F)}(\tau_1 \boxtimes \tau_2) \simeq \mathrm{ind}_{P^0(m)}^{\mathrm{GL}_4(\mathcal{O}_F)}(\lambda_1 \boxtimes \lambda_2)$$

for all $m \geq N$.

By methods similar to those of chapters 2,3,4,5 we will reduce the problem of classifying typical representations for s to finding typical representations occurring as sub-representations of

$$\mathrm{ind}_{P^0(N+1)}^{\mathrm{GL}_4(\mathcal{O}_F)}(\lambda_1 \boxtimes \lambda_2). \tag{1.7}$$

It turns out that the Bushnell-Kutzko type for s is not of the form $(P^0(N+1), \lambda_1 \boxtimes \lambda_2)$ for almost all cases. This means that we cannot directly conclude that typical representations for the inertial class s are precisely the sub-representations of (1.7).

We expect (at least when $\#k_F > 2$) that typical representations must occur as sub-representation of

$$\mathrm{ind}_{J_s}^{\mathrm{GL}_4(\mathcal{O}_F)}(\lambda_1 \boxtimes \lambda_2).$$

We compared the dimensions of the above representation with that of (1.7) and observed that they are indeed different and this gave us the first heuristic to expect that there are non-typical irreducible sub-representations occurring in (1.7). We tried and indeed succeed in many cases to classify typical representations occurring in (1.7). This is the a new feature we show in this chapter. We can explain the main result by first recalling some aspects of the construction of Bushnell-Kutzko (semi-simple) type for s.

Let $[\mathfrak{A}_i, n_i, 0, \beta_i]$ be a simple strata defining the Bushnell-Kutzko type (J_i^0, λ_i) for $i \in 1, 2$. We denote by e_1 and e_2 the ramification indices of the orders $\mathfrak{A}_1$ and $\mathfrak{A}_2$ respectively. We denote by ϕ_i the irreducible factor of the characteristic polynomial associated to the simple stratas $[\mathfrak{A}_i, n_i, 0, \beta_i]$ for $i \in \{1, 2\}$ (see [BK93][Section 2.3]). We have broadly two cases.

1. $n_1/e_1 \neq n_2/e_2$; $n_1/e_1 = n_2/e_2$ but $\phi_1 \neq \phi_2$.

2. $n_1/e_1 = n_2/e_2$ and $\phi_1 = \phi_2$

The representations σ_1 and σ_2 are called **completely distinct** if they satisfy condition (1). Otherwise they are said to have **common approximation**. The case of common approximation can be divided into 2 cases. The first is called homogeneous case i.e common approximation to level zero. The homogeneous case in our present situation (i.e both σ_1 and σ_2 are representations of $\mathrm{GL}_2(F)$) means that $\mathfrak{A}_1 = \mathfrak{A}_2 := \mathfrak{A}$, $n_1 = n_2 := n$ and $\beta_1 = \beta_2 := \beta$. And the simple character defining the β extension κ is also the same for σ_1 and σ_2. The second case is common approximation to level $l > 0$. Due to lack of time we do not treat the case where σ_1 and σ_2 admit common approximation to level $l > 0$. Our main theorem is:

Theorem 1.3.5. *Let $\#k_F > 3$ and s be the inertial class*

$$[\mathrm{GL}_2(F) \times \mathrm{GL}_2(F), \sigma_1 \boxtimes \sigma_2]$$

where σ_1 and σ_2 are completely distinct or homogenous. The typical representations for the inertial class s are precisely the irreducible sub-representations of

$$\mathrm{ind}_{J_s}^{\mathrm{GL}_4(\mathcal{O}_F)}(\lambda_s).$$

We will first consider the classification of typical representations for the homogeneous case. We sketch the proof in some detail:

The Bushnell-Kutzko's type (J_s, λ_s) for the inertial class $s = [M, \sigma_1 \boxtimes \sigma_2]$ with each σ_i containing type (J^0, λ_i) (which is defined by the simple strata $[\mathfrak{A}, n, 0, \beta]$) is given by $\lambda_s = \lambda_1 \boxtimes \lambda_2$ and

$$J_s := \begin{pmatrix} J^0 & \mathcal{O}_E + \mathfrak{P}_{\mathfrak{A}}^{(n-t)} \\ \mathfrak{P}_E + \mathfrak{P}_{\mathfrak{A}}^{t+1} & J^0 \end{pmatrix}$$

for $t = [n/2]$ and $E = F[\beta]$. One important observation is that $J_s \cap U \neq U(\mathcal{O}_F)$. This gives us the main problem. To tackle this situation we first try to modify the induced representation (1.7) at the level of subgroups close to the $P^0(n+1)$ and J_s. We will explain this in the unramified ($e(E|F) = 1$) case since most of notations are already defined.

The first step is to split the representation

$$\rho_1 := \mathrm{ind}_{P^0(n+1)}^{P^0(t+1)}(\lambda_s).$$

The idea is to replace the group $P^0(n+1)$ with the group J_s and see the intertwining between ρ_1 and

$$\rho_2 := \mathrm{ind}_{J_s}^{P^0(t+1)}(\lambda_s).$$

But we cannot do this for the very basic reason that $P^0(t)$ does not contain the group J_s. Now we use a smaller group J_s' such that J_s' is contained in the group $P^0(t)$. The only change between the J_s' and J_s is their intersection with the lower unipotent group. This group J_s' has the property that $P^0(n+1)J_s' = P^0(t)$ and the induction

$$\rho_3 := \mathrm{ind}_{J_s'}^{P^0(t)}(\lambda_s)$$

is irreducible. We are in good situation since by Mackey decomposition the space of intertwining operators between ρ_1 and ρ_3 are one dimensional and any non-zero intertwining operator is surjective from the irreducibility of ρ_3. The rest of the proof is showing that the kernel of this non-trivial intertwining operator has irreducible sub-representations which also occur in ρ_1 for some suitably modified λ_1 and λ_2. The proof of this will now include the action of the group $U(\mathcal{O}_F)$. Which so far acts trivially on such inductions.

The non-trivial intertwining operator I between ρ_1 and ρ_3 is given by the following integral:

$$I(f)(p) = \int_{u^-\in\ P(s,t)\cap\bar{U}} f(u^-p)du^-.$$

If a function f is in the kernel of I then we have

$$\int_{u^-\in\ P^0(t)\cap\bar{U}} f(u^-u^+(u^-)^{-1}u^-)du^- = 0$$

for all $u^+ \in P^0(t)$. Let u^- and u^+ be represented in 2×2 block matrices as

$$u^- = \begin{pmatrix} \mathrm{id} & 0 \\ U^- & \mathrm{id} \end{pmatrix},\ u^+ = \begin{pmatrix} \mathrm{id} & U^+ \\ 0 & \mathrm{id} \end{pmatrix}$$

respectively. The above integral-equation can be written as

$$\int_{u^-\in\ P^0(t)\cap\bar{U}} \psi_{(\beta U^+ - U^+\beta)}(1+U^-)f(u^-)du^- = 0. \tag{1.8}$$

We first note that the group of characters of the group

$$P^0(t+1)/P^0(n+1) \simeq (P^0(t+1)\cap\bar{U})/(P^0(n+1)\cap\bar{U}) \simeq \mathfrak{P}_{\mathfrak{A}}^{t+1}/\mathfrak{P}_{\mathfrak{A}}^{n+1}$$

is isomorphic to $\mathfrak{P}_{\mathfrak{A}}^{-n}/\mathfrak{P}_{\mathfrak{A}}^{-t}$. The kernel of I is spanned by the characters which are not in the image of $[\beta,\cdot]$ (the commutator bracket with β on $\mathfrak{P}_{\mathfrak{A}}^{-n}/\mathfrak{P}_{\mathfrak{A}}^{-t}$).

Let W be a sub-representation of $\ker(I)$. We will use the fact in lemma 6.2.3 show that irreducible sub-representations of

$$\mathrm{ind}_{P^0(t)}^{\mathrm{GL}_4(\mathcal{O}_F)}(W)$$

are non-typical with the help of co-restriction map.

Now we are left to understand the representation

$$\mathrm{ind}_{J_s'}^{\mathrm{GL}_4(\mathcal{O}_F)}(\lambda_s).$$

The group J_s' is contained in J_s and by familiar transitivity and pull-back push-forward technique we write

$$\mathrm{ind}_{J_s'}^{\mathrm{GL}_4(\mathcal{O}_F)}(\lambda_s) \simeq \mathrm{ind}_{J_s}^{\mathrm{GL}_4(\mathcal{O}_F)}\{\mathrm{ind}_{J_s'}^{J_s}(\mathrm{id}) \otimes (\lambda) \simeq \mathrm{ind}_{J_s}^{\mathrm{GL}_4(\mathcal{O}_F)}(\lambda_s) \oplus \Gamma$$

and we show that irreducible sub-representations of Γ are non-typical. Which shows that typical representations are precisely the irreducible sub-representations of the first summand in the above decomposition.

We point out that the proof of the facts about the kernel of the intertwining operator we introduced earlier depends on the exact-sequence machinery of Bushnell-Kutzko. The surjectivity of the operator also follows from the calculation of the sets $\mathfrak{N}(\mathfrak{A}, \beta)$. For the ramified case $(e(E|F) > 1)$ we will meet a situation to calculate $\mathfrak{N}(\mathfrak{A}, \beta)$ when $E = F[\beta]$ does not normalize the order $\mathfrak{A}$. This requires some attention otherwise we can use the Bushnell-Kutzko machinery to complete the classification.

Now we have to treat the case where σ_1 and σ_2 are completely distinct. A careful modification for the representation

$$\mathrm{ind}_{P^0(N+1)}^{\mathrm{GL}_n(\mathcal{O}_F)}(\lambda_1 \boxtimes \lambda_s)$$

gives the theorem in the case where σ_1 and σ_2 are completely distinct. We will use techniques similar to those developed to treat the homogenous case. This is the reason we choose to put this rather simpler situation at the end.

Chapter 2

Preliminaries

The following notation will be used in all chapters of this thesis.

2.1 Basic notation

Let F be a non-Archimedean local field with ring of integers $\mathcal{O}_F$, maximal ideal $\mathfrak{P}_F$ and a finite residue field k_F. We denote by G the F-rational points of an algebraic reductive group and by P a F-rational parabolic subgroup of G. All our representations are on vector spaces over $\mathbb{C}$. Let σ be a smooth representation of a Levi-subgroup M of P. We denote by $i_P^G(\sigma)$ the normalized parabolically-induced representation.

Let H be a closed subgroup of G and (τ, V) be a smooth representation of H, we denote by $\mathrm{ind}_H^G(\tau)$ the space of functions $f : G \to V$ such that $f(hg) = \tau(h)f(g)$ and there exists a compact open subgroup K_f of G such that $f(gk) = f(g)$ for all $g \in G$ and $k \in K_f$. The group G acts on these functions by right multiplication i.e $(g'f)(g) = f(gg')$ for all $g, g' \in G$ and $f \in \mathrm{ind}_H^G(\tau)$. We denote by c-$\mathrm{ind}_H^G(\tau)$ the sub-space of $\mathrm{ind}_H^G(\tau)$ consisting of functions f such that $\sup(f) \subset HX_f$ where X_f is a compact set. This is the compactly induced representation.

Let H_1 and H_2 be two subgroups such that $H_2 \subset H_1$, σ be a representation of H_1 we denote by $\mathrm{res}_{H_2}(\sigma)$ the restriction of σ to H_2. We use $\boxtimes$ and $\otimes$ for the tensor product of representations of two different groups and the same group respectively. If H_2 is a subgroup of a group H_1, τ is a representation of H_2 and $h \in H_1$ then we denote by τ^h the representation of hH_2h^{-1} given by $h' \mapsto \tau(h^{-1}h'h)$ for all $h' \in hH_2h^{-1}$.

After recalling some general definitions we will restrict ourself to the case $G = \mathrm{GL}_n(F)$ and we will use the following notation: Let $I = (n_1, n_2, n_3, \ldots, n_r)$ be an ordered partition of a positive integer n. Let P_I be the group of invertible block upper triangular matrices of the type $(n_1, n_2, \ldots, n_r)$. We denote by M_I and U_I the group of block diagonal matrices of the type I and the unipotent radical of P_I respectively. We call P_I and M_I the standard parabolic subgroup and standard Levi-subgroup of type I respectively. We denote by $K_n(m)$ the principal congruence subgroup of $\mathrm{GL}_n(\mathcal{O}_F)$ of level m.

2.2 Bernstein decomposition and typical representations

Let $B(G)$ be the set of pairs (M, σ) where M is a Levi-subgroup of a F-parabolic subgroup P of G and σ is an irreducible supercuspidal representation of M. We define an equivalence relation on $B(G)$ by setting

$$(M_1, \sigma_1) \sim (M_2, \sigma_2)$$

if and only if there exists an element $g \in G$ and an unramified character χ of M_2 such that $M_1 = gM_2g^{-1}$ and $\sigma_1^g \simeq \sigma_2 \otimes \chi$. We denote by $\mathcal{B}_G$ the set of such equivalence classes called **inertial classes or Bernstein components**. Any irreducible smooth representation π of G occurs as a sub-representation of a parabolic induction $i_P^G(\sigma)$ where σ is an irreducible supercuspidal representation of a Levi-subgroup M of P. The pair (M, σ) is well determined up to G-conjugation. We call the class $s = [M, \sigma]$ the **inertial support of** π. Let $\mathcal{M}(G)$ be the category of all smooth representations of G. For an inertial class $s = [M, \sigma]$ we denote by $\mathcal{M}_s(G)$ the full sub-category consisting of smooth representations all of whose irreducible sub-quotients appear in the composition series of $i_P^G(\sigma \otimes \chi)$ where χ is an unramified character of M. It is shown by Bernstein (see [Ren10][VI.7.2, Theorem]) that the category $\mathcal{M}(G)$ decomposes as a direct product of $\mathcal{M}_s(G)$ in particular every smooth representation can be written as a direct sum of objects in the categories $\mathcal{M}_s(G)$. We denote by $\mathcal{A}_G(s)$ the set of isomorphism classes of simple objects in the category $\mathcal{M}_s(G)$. If $G = \mathrm{GL}_n(F)$ we use the notation $\mathcal{A}_n(s)$ for $\mathcal{A}_G(s)$ and $\mathcal{B}_n$ for $\mathcal{B}_{\mathrm{GL}_n(F)}$.

Given an irreducible smooth representation ρ of a maximal compact subgroup K of G the compact induction $\pi := \text{c-ind}_K^G(\rho)$ is a finitely generated smooth representation of G and hence there exists an irreducible G-quotient of π. By Frobenius reciprocity [BH06, Proposition 2.5] we get that ρ occurs in a smooth irreducible representation of G. For a given inertial class, we are interested in the representations ρ of K which only occur in irreducible smooth representations with inertial support s.

Definition 2.2.1. *Let s be an inertial class for G. An irreducible smooth representation τ of a maximal compact subgroup K of G is called K-typical representation for s if for any irreducible smooth representation π of G, $\mathrm{Hom}_K(\tau, \pi) \neq 0$ implies that $\pi \in \mathcal{A}_G(s)$.*

In this thesis we will confine ourself to the case where $G = \mathrm{GL}_n(F)$, $K = \mathrm{GL}_n(\mathcal{O}_F)$ and $n \geq 2$ and in this case we call a K-typical representation for s a typical representation for s. An irreducible representation τ of $\mathrm{GL}_n(\mathcal{O}_F)$ is called *atypical* if τ occurs in two smooth representations π_1 from $\mathcal{M}_s(\mathrm{GL}_n(F))$ and π_2 from $\mathcal{M}_{s'}(\mathrm{GL}_n(F))$ such that $s \neq s'$.

For any component $s \in \mathcal{B}_n$, the existence of a typical representation can be deduced from the theory of types developed by Bushnell and Kutzko in the articles [BK99] and [BK93]. Bushnell and Kutzko constructed a pair (J_s, λ_s) where J_s is a compact open subgroup of $\mathrm{GL}_n(F)$ and λ_s is an irreducible representation of J_s. Let π be an irreducible smooth representation of $\mathrm{GL}_n(F)$. The pair (J_s, λ_s) satisfies the condition

$$\mathrm{Hom}_{J_s}(\pi, \lambda_s) \neq 0 \;\Leftrightarrow\; \pi \;\in\; \mathcal{A}_n(s).$$

In the case of $\mathrm{GL}_n(F)$, the group J_s can be arranged to be a subgroup of $\mathrm{GL}_n(\mathcal{O}_F)$ by conjugating with an element of $\mathrm{GL}_n(F)$ and hence we assume that $J_s \subset \mathrm{GL}_n(\mathcal{O}_F)$. It follows from Frobenius reciprocity that any irreducible sub-representation of

$$\mathrm{ind}_{J_s}^{\mathrm{GL}_n(\mathcal{O}_F)}(\lambda_s) \tag{2.1}$$

is a typical representation. The irreducible sub-representations of (2.1) are classified by Schneider and Zink in [SZ99, Section 6, $T_{K,\lambda}$ functor].

For $s = [\mathrm{GL}_n(F), \sigma]$, Paskunas in the article [Pas05][Theorem 8.1] showed that up to isomorphism there exists a unique typical representation for s. More precisely,

Theorem 2.2.2 (Paskunas)**.** *Let n be a positive integer greater than one and σ be an irreducible supercuspidal representation of $\mathrm{GL}_n(F)$. Let (J_s, λ_s) be a Bushnell-Kutzko type for the component $s = [\mathrm{GL}_n(F), \sigma]$ with $J_s \subset \mathrm{GL}_n(\mathcal{O}_F)$. The representation*

$$\mathrm{ind}_{J_s}^{\mathrm{GL}_n(\mathcal{O}_F)}(\lambda_s)$$

is the unique typical representation for the component $[\mathrm{GL}_n(F), \sigma]$ and occurs with multiplicity one in $\sigma \otimes \chi$ for all unramified characters χ of $\mathrm{GL}_n(F)$.

We will consider the classification of typical representations for components $[M, \sigma]$ where M is a Levi-subgroup of a proper parabolic subgroup of $\mathrm{GL}_n(F)$.

Let $s = [M, \sigma]$ be an inertial class of $\mathrm{GL}_n(F)$. We will choose a representative for s. Let P be a parabolic subgroup with M a its Levi-factor. There exists a $g \in \mathrm{GL}_n(F)$ such that $gPg^{-1} = P_I$ for some ordered partition $I = (n_1, n_2, \ldots, n_r)$ of n. The groups gMg^{-1} and M_I are two Levi-factors of P_I hence we get an $u \in \mathrm{Rad}\, P_I$ such that $ugM(ug)^{-1} = M_I$. This shows that there exists an element $g' \in \mathrm{GL}_n(F)$ such that $g'Mg'^{-1} = M_I$. Let J be a permutation of the ordered partition $(n_1, n_2, \ldots, n_r)$. We can choose a $g'' \in \mathrm{GL}_n(F)$ such that M_I and M_J are conjugate, the two pairs (M, σ) and $(M_J, \sigma^{g'g''})$ are inertially equivalent. In certain cases it is convenient to choose a particular permutation. For example in the proof of the main theorem of chapter 3 we choose $J = (n_1', n_2', \ldots n_r')$ such that $n_i' \leq n_j'$ for all $i \leq j$. We denote by σ_I and σ_J the representations $\sigma^{g'}$ and $\sigma^{g'g''}$ respectively and hence

$s = [M_I, \sigma_I] = [M_J, \sigma_J]$.

Let τ be a typical representation for the component s. The representation τ occurs as a $\mathrm{GL}_n(\mathcal{O}_F)$ sub-representation of a $\mathrm{GL}_n(F)$-irreducible smooth representation π (see the reasoning given in the paragraph above Definition 2.2.1). From the above paragraph π occurs in the composition series of $i_{P_I}^{\mathrm{GL}_n(F)}(\sigma_I)$ where σ_I is a supercuspidal representation of M_I. Hence to classify typical representations we fix a pair $(M_I, \sigma_I) \sim (M, \sigma)$ and examine the $\mathrm{GL}_n(\mathcal{O}_F)$-irreducible sub-representations of

$$\mathrm{res}_{\mathrm{GL}_n(\mathcal{O}_F)}(i_{P_I}^{\mathrm{GL}_n(F)}(\sigma_I)),$$

looking for possible typical representations for s.

By the Iwasawa decomposition $\mathrm{GL}_n(F) = \mathrm{GL}_n(\mathcal{O}_F)P_I$ we get that

$$\mathrm{res}_{\mathrm{GL}_n(\mathcal{O}_F)}(i_{P_I}^{\mathrm{GL}_n(F)}(\sigma_I)) \simeq \mathrm{ind}_{P_I \cap \mathrm{GL}_n(\mathcal{O}_F)}^{\mathrm{GL}_n(\mathcal{O}_F)}(\sigma_I).$$

We write σ_I as $\boxtimes_{i=1}^r \sigma_i$ where σ_i is a supercuspidal representation of $\mathrm{GL}_{n_i}(F)$ for $1 \leq i \leq r$. We denote by τ_i the unique typical representation for the component $[\mathrm{GL}_{n_i}(F), \sigma_i]$ for $1 \leq i \leq r$ and let τ_I be the $M_I(\mathcal{O}_F)$-representation $\boxtimes_{i=1}^r \tau_i$. Will Conley observed in his thesis that the representation

$$\mathrm{ind}_{P_I \cap \mathrm{GL}_n(\mathcal{O}_F)}^{\mathrm{GL}_n(\mathcal{O}_F)}(\tau_I)$$

admits a complement in $\mathrm{ind}_{P_I \cap \mathrm{GL}_n(\mathcal{O}_F)}^{\mathrm{GL}_n(\mathcal{O}_F)}(\sigma_I)$ whose irreducible sub-representations are atypical for s. We prove a mild generalization which will be used later in proofs by induction.

Let $t_i = [M_i, \lambda_i]$ be a Bernstein component of $\mathrm{GL}_{n_i}(F)$ for $1 \leq i \leq r$. Let σ_i be a smooth representation from $\mathcal{M}_{t_i}(\mathrm{GL}_{n_i}(F))$. We suppose $\mathrm{res}_{\mathrm{GL}_{n_i}(\mathcal{O}_F)}\, \sigma_i = \tau_i^0 \oplus \tau_i^1$ for $1 \leq i \leq r$ such that irreducible sub-representations of τ_i^1 are atypical. We denote by t the Bernstein component

$$[M_1 \times M_2 \times \cdots \times M_r, \lambda_1 \boxtimes \lambda_2 \boxtimes \cdots \boxtimes \lambda_r]$$

of $\mathrm{GL}_n(F)$. The component t is independent of the choice of representatives (M_i, λ_i). Let $\tau_I^0 = \boxtimes_{i=1}^r \tau_i^0$ and $\sigma_I = \boxtimes_{i=1}^r (\sigma_i)$.

Proposition 2.2.3. *The representation*

$$\mathrm{ind}_{P_I \cap \mathrm{GL}_n(\mathcal{O}_F)}^{\mathrm{GL}_n(\mathcal{O}_F)}(\tau_I^0)$$

admits a complement in $i_{P_I}^{\mathrm{GL}_n(F)}(\sigma_I)$ *with all its irreducible sub-representations atypical for* t.

Proof. Any $\mathrm{GL}_n(\mathcal{O}_F)$-irreducible sub-representation of $i_{P_I}^{\mathrm{GL}_n(F)}(\sigma_I)$ occurs as a sub-representation of

$$\mathrm{ind}_{P_I\cap \mathrm{GL}_n(\mathcal{O}_F)}^{\mathrm{GL}_n(\mathcal{O}_F)}(\boxtimes_{i=1}^r \gamma_i) \tag{2.2}$$

where γ_i is a $\mathrm{GL}_{n_i}(\mathcal{O}_F)$-irreducible sub-representation of σ_i. If γ_i occurs in τ_i^1 for some $i = N$ with $N \leq r$ then there exists a component $t'_N \in \mathcal{B}_N$ such that $t'_N = [M'_N, \lambda'_N] \neq t_N$ and γ_N occurs in the restriction $\mathrm{res}_{\mathrm{GL}_n(\mathcal{O}_F)}\, i_{P'_N}^{\mathrm{GL}_{n_N}(F)}(\lambda'_N)$. Hence the representation (2.2) occurs as a $\mathrm{GL}_n(\mathcal{O}_F)$-sub-representation of

$$i_P^{\mathrm{GL}_n(F)}\{i_{P_1}^{\mathrm{GL}_{n_1}(F)}(\lambda_1)\boxtimes\cdots\boxtimes i_{P'_N}^{\mathrm{GL}_{n_N}(F)}(\lambda'_N)\boxtimes\cdots\boxtimes i_{P_r}^{\mathrm{GL}_{n_r}(F)}(\lambda_r)\}$$

The inertial support t' of the above representation is

$$[M_1\times\cdots\times M'_N\times\cdots\times M_r, \lambda_1\boxtimes\cdots\boxtimes\lambda'_N\boxtimes\cdots\boxtimes\lambda_r].$$

We may assume that M_i is a standard Levi-subgroup for $1 \leq i \leq r$. Now

$$[M_N = \prod_{j=1}^{p}\mathrm{GL}_{m_j}(F), \lambda_N = \boxtimes_{j=1}^p \zeta_j] \neq [M'_N = \prod_{j=1}^{p'}\mathrm{GL}_{m'_j}(F), \lambda'_N = \boxtimes_{j=1}^{p'}\zeta'_j]$$

implies that there exists a cuspidal component $[\mathrm{GL}_{m_k}(F), \zeta_k]$ occurring in the multi-set

$$\{[\mathrm{GL}_{m_1}(F),\zeta_1],[\mathrm{GL}_{m_2}(F),\zeta_2],\dots,[\mathrm{GL}_{m_p}(F),\zeta_p]\}$$

which has a different multiplicity in

$$\{[\mathrm{GL}_{m'_1}(F),\zeta'_1],[\mathrm{GL}_{m'_2}(F),\zeta'_2],\dots,[\mathrm{GL}_{m_{p'}}(F),\zeta'_{p'}]\}.$$

Adding cuspidal components with the same multiplicity to the above two multi-sets cannot make the multiplicities of the component $[\mathrm{GL}_k(F), \zeta_k]$ the same. This shows that $t' \neq t$ and hence the desired complement is the direct sum of the representations as in (2.2) such that γ_i occur in τ_i^1 for some $i \in \{1, 2, \dots, r\}$. $\square$

Lemma 2.2.4. *Let $t_i = [\mathrm{GL}_{n_i}(F), \sigma_i]$ be a Bernstein component for $\mathrm{GL}_{n_i}(F)$ and τ_i be a typical representation for t_i. The representation*

$$\mathrm{ind}_{P_I\cap \mathrm{GL}_n(\mathcal{O}_F)}^{\mathrm{GL}_n(\mathcal{O}_F)}(\tau_I)$$

admits a complement in $i_{P_I}^{\mathrm{GL}_n(F)}(\sigma_I)$ whose irreducible sub-representations are atypical.

Proof. We use the uniqueness of typical representations for supercuspidal representations (see [Pas05]) to decompose $\mathrm{res}_{\mathrm{GL}_n(\mathcal{O}_F)}\,\sigma_i$ as $\tau_i \oplus \tau_i^1$ such that irreducible sub-representations of τ_i^1 are atypical. The lemma follows as a consequence of proposition 2.2.3. $\square$

Given a component $s = [M_I, \sigma_I]$ of $\mathrm{GL}_n(F)$ the above lemma shows that typical representations only occur as sub-representations of

$$\mathrm{ind}_{P_I \cap \mathrm{GL}_n(\mathcal{O}_F)}^{\mathrm{GL}_n(\mathcal{O}_F)}(\tau_I)$$

The above representation is still an infinite dimensional representation of the compact group $\mathrm{GL}_n(\mathcal{O}_F)$. We write the above representation as an increasing union of finite dimensional representations.

Let $\{H_i\}_{i\geq 1}$ be a decreasing sequence of compact open subgroups of the maximal compact subgroup $\mathrm{GL}_n(\mathcal{O}_F)$. Let $\bar{U}_I$ be the unipotent radical of the opposite parabolic subgroup $\bar{P}_I$ of P_I with respect to the Levi-subgroup M_I. We assume that H_i satisfies Iwahori decomposition with respect to the parabolic subgroup P_I and Levi-subgroup M_I for all $i \geq 1$ i.e. the product map

$$(H_i \cap \bar{U}_I) \times (H_i \cap M_I) \times (H_i \cap U_I) \to H_i$$

is a homeomorphism for any ordering of the factors on the left hand side and that $\bigcap_{i\geq 1} H_i = \mathrm{GL}_n(\mathcal{O}_F) \cap P_I$. Let τ be a finite dimensional smooth representation of the group $M_I(\mathcal{O}_F)$. We assume that τ extends to a representation of H_i for all $i \geq 1$ such that $H_i \cap U_I$ and $H_i \cap \bar{U}_I$ are contained in the kernel of τ. By definition the representation $\mathrm{ind}_{H_i}^{\mathrm{GL}_n(\mathcal{O}_F)}(\tau)$ is contained in $\mathrm{ind}_{\mathrm{GL}_n(\mathcal{O}_F)\cap P_I}^{\mathrm{GL}_n(\mathcal{O}_F)}(\tau)$.

Lemma 2.2.5. *The union of the representations*

$$\mathrm{ind}_{H_i}^{\mathrm{GL}_n(\mathcal{O}_F)}(\tau)$$

for all $i \geq 1$ is equal to the representation

$$\mathrm{ind}_{\mathrm{GL}_n(\mathcal{O}_F)\cap P_I}^{\mathrm{GL}_n(\mathcal{O}_F)}(\tau).$$

Proof. Let W be the underlying space for the representations τ. Any element f on in the space

$$\mathrm{ind}_{\mathrm{GL}_n(\mathcal{O}_F)\cap P_I}^{\mathrm{GL}_n(\mathcal{O}_F)}(\tau)$$

is a function $f : \mathrm{GL}_n(\mathcal{O}_F) \to W$ such that

1. $f(pk) = \tau(p)f(k)$ for all $p \in \mathrm{GL}_n(\mathcal{O}_F) \cap P_I$ and $k \in \mathrm{GL}_n(\mathcal{O}_F)$,

2. There exists a positive integer m (depending on f) such that $f(gk) = f(g)$ for all $k \in K_n(m)$ and $g \in \mathrm{GL}_n(\mathcal{O}_F)$.

Now there exists a positive integer i such that $H_i \cap \bar{U} \subset K_n(m)$. For such a choice of i and $h \in H_i$ write $h = h^- h^+$ where $h^+ \in \mathrm{GL}_n(\mathcal{O}_F) \cap P$, $h^- \in H_i \cap \bar{U}$ and we can do so by Iwahori decomposition of H_i. We observe that $f(hk) = f(h^-h^+k) = f(h^+k(h^+k)^{-1}h^-(h^+k)) = f(h^+k) = \tau(h^+)f(k)$ (since $(h^+k)^{-1}h^-(h^+k) \in K_n(m)$). Hence $f \in \mathrm{ind}_{H_i}^{\mathrm{GL}_n(\mathcal{O}_F)}(\tau)$. $\square$

We shall need the following technical lemma for frequent reference. Let P be any parabolic subgroup of $\mathrm{GL}_n(F)$ with a Levi-subgroup M and U be the unipotent radical of P. Let J_1 and J_2 be two compact open sub-groups of $\mathrm{GL}_n(\mathcal{O}_F)$ such that J_1 contains J_2. Suppose J_1 and J_2 both satisfy Iwahori decomposition with respect to the Levi-subgroup M, $J_1 \cap U = J_2 \cap U$ and $J_1 \cap \bar{U} = J_2 \cap \bar{U}$. Let λ is an irreducible smooth representation of J_2 which admits an Iwahori decomposition i.e. $J_2 \cap U$ and $J_2 \cap \bar{U}$ are contained in the kernel of λ.

Lemma 2.2.6. *The representation $\mathrm{ind}_{J_2}^{J_1}(\lambda)$ is the extension of the representation $\mathrm{ind}_{J_2\cap M}^{J_1\cap M}(\lambda)$ such that $J_1 \cap U$ and $J_1 \cap \bar{U}$ are contained in the kernel of the extension.*

Proof. From Iwahori decomposition we get that $(J_1 \cap M)J_2 = J_1$ and Mackey decomposition we get that

$$\mathrm{res}_{J_1\cap M} \mathrm{ind}_{J_2}^{J_1}(\lambda) \simeq \mathrm{ind}_{J_2\cap M}^{J_1\cap M}(\lambda).$$

We now verify that $J_1 \cap U$ and $J_1 \cap \bar{U}$ act trivially on $\mathrm{ind}_{J_2}^{J_1}(\lambda)$. Observe that

$$\mathrm{res}_{J_1\cap P} \mathrm{ind}_{J_2}^{J_1}(\lambda) \simeq \mathrm{ind}_{J_2\cap P}^{J_1\cap P}(\lambda).$$

Since the double coset representatives for

$$\frac{J_1 \cap P}{J_2 \cap P}$$

can be chosen from $M \cap J_1$ the group $J_1 \cap U$ acts trivially on $\mathrm{ind}_{J_2}^{J_1}(\lambda)$. Similarly $J_1 \cap \bar{U}$ acts trivially on $\mathrm{ind}_{J_2}^{J_1}(\lambda)$. This concludes the lemma. $\square$

Lemma 2.2.7. *Let G be the F-rational points of an algebraic reductive group and χ be a character of G. Let τ be a K-typical representation for the component $s = [M, \sigma]$. The representation $\tau \otimes \chi$ is a typical representation for the component $[M, \sigma \otimes \chi]$.*

Proof. Let $\mathrm{Hom}_K(\tau \otimes \chi, \pi) \neq 0$ for some irreducible smooth representation π of G. We now have $\mathrm{Hom}_K(\tau, \pi \otimes \chi^{-1}) \neq 0$. This implies that $\pi \otimes \chi^{-1}$ occurs in the composition series of

$$i_P^G(\sigma \otimes \eta)$$

for some parabolic subgroup P containing M is a Levi-factor and η an unramified character of M. Now π occurs in the composition series for the representation

$$i_P^G(\sigma \otimes \chi \otimes \eta)$$

hence $\tau \otimes \chi$ is a K-typical representation for the component $[M, \sigma \otimes \chi]$. $\square$

We now sketch the general strategy to classify typical representations. We choose the sequence $\{H_i \mid i \geq 1\}$ depending on certain class of components s and then we will construct a representation say $U_i(\tau_I)$ such that

$$U_i(\tau_I) \oplus \mathrm{ind}_{H_1}^{\mathrm{GL}_n(\mathcal{O}_F)}(\tau_I) = \mathrm{ind}_{H_i}^{\mathrm{GL}_n(\mathcal{O}_F)}(\tau_I)$$

for $i \geq 2$. We will show by induction on the integer i that the irreducible sub-representations of $U_i(\tau_I)$ are atypical for s. This shows that typical representations occur as sub-representations of

$$\mathrm{ind}_{H_1}^{\mathrm{GL}_n(\mathcal{O}_F)}(\tau_I). \tag{2.3}$$

It is indeed possible that all irreducible sub-representations of (2.3) are typical for s. This will be the case for many Bernstein components for example level-zero (to be defined in the next chapter), principal series components and $s = [\mathrm{GL}_{n-1}(F) \times \mathrm{GL}_1(F), \sigma \boxtimes \chi]$ for $n \geq 2$. The choice of H_i, construction of $U_i(\tau_I)$ and showing irreducible sub-representations of $U_i(\tau_I)$ are atypical representations for various classes of components will occupy the next three chapters.

Chapter 3

Level zero inertial classes

Definition 3.0.8. *Let $I = (n_1, n_2, \ldots, n_r)$ be an ordered partition of n. An inertial class $s = [M_I, \boxtimes_{i=1}^r \sigma_i]$ is called a level-zero inertial class if for every σ_i there exists an irreducible representation τ_i of $\mathrm{GL}_{n_i}(\mathcal{O}_F)$ such that τ_i is the inflation of an irreducible cuspidal representation of $\mathrm{GL}_{n_i}(k_F)$ and $\mathrm{Hom}_{\mathrm{GL}_{n_i}(\mathcal{O}_F)}(\tau_i, \sigma_i) \neq 0$.*

We fix a level-zero inertial class $s = [M_I, \sigma_I]$ with the pairs $(\mathrm{GL}_{n_i}(\mathcal{O}_F), \tau_i)$ as in the above definition. **The pair $(\mathrm{GL}_{n_i}(\mathcal{O}_F), \tau_i)$ is the Bushnell-Kutzko type for the inertial class** $[\mathrm{GL}_{n_i}(F), \sigma_i]$. Let m be a positive integer and $P_I(m)$ be the inverse image of $P_I(\mathcal{O}_F/\mathfrak{P}_F^m)$ under the mod-$\mathfrak{P}_F^m$ reduction map

$$\pi_m : \mathrm{GL}_n(\mathcal{O}_F) \to \mathrm{GL}_n(\mathcal{O}_F/\mathfrak{P}_F^m).$$

The representation $\boxtimes_{i=1}^r \tau_i$ of $M_I(k_F)$ extends to a representation of $P_I(k_F)$ by inflation via the quotient map

$$P_I(k_F) \to P_I(k_F)/U_I(k_F) \simeq M_I(k_F).$$

The representation $\boxtimes_{i=1}^r \tau_i$ of $P_I(k_F)$ extends to a representation of $P_I(1)$ by inflation via the map π_1. We note that $P_I(1) \cap U_I$ and $P_I(1) \cap \bar{U}_I$ are contained in the kernel of this extension. The pair $(P_I(1), \tau_I)$ is the Bushnell-Kutzko type for the component s (see [BK99][Section 8.3.1]). The irreducible sub-representations of

$$\mathrm{ind}_{P_I(1)}^{\mathrm{GL}_n(\mathcal{O}_F)}(\tau_I)$$

are thus typical for s.

We note that the groups $P_I(m)$ satisfy Iwahori decomposition with respect to P_I and M_I. The $M_I(\mathcal{O}_F)$ representation τ_I extends to a representation of $P_I(m)$ such that $P_I(m) \cap U_I$ and $P_i(m) \cap \bar{U}_I$ are contained in the kernel of the extension. This shows that the sequence of groups $\{P_I(m) \mid m \geq 1\}$ and τ_I satisfy the hypothesis for the groups $\{H_m \mid m \geq 1\}$ and τ in lemma 2.2.5 hence we have the isomorphism

$$\bigcup_{m \geq 1} \mathrm{ind}_{P_I(m)}^{\mathrm{GL}_n(\mathcal{O}_F)}(\tau_I) \simeq \mathrm{ind}_{P_I \cap \mathrm{GL}_n(\mathcal{O}_F)}^{\mathrm{GL}_n(\mathcal{O}_F)}(\tau_I).$$

We recall that the lemma 2.2.4 shows that typical representations for the component s can only occur in the above representation.

Using Frobenius reciprocity we get that the representation τ_I occurs in $\mathrm{ind}_{P_I(m)}^{P_I(1)}(\tau_I)$ with multiplicity one. Let $m \geq 1$ and $U_m^0(\tau_I)$ be the $P_I(1)$-stable complement of the representation τ_I in $\mathrm{ind}_{P_I(m)}^{P_I(1)}(\tau_I)$. Let $U_m(\tau_I)$ be the representation

$$\mathrm{ind}_{P_I(1)}^{\mathrm{GL}_n(\mathcal{O}_F)}(U_m^0(\tau_I)).$$

We note that

$$\mathrm{ind}_{P_I(1)}^{\mathrm{GL}_n(\mathcal{O}_F)}(\tau_I) \oplus U_m(\tau_I) \simeq \mathrm{ind}_{P_I(m)}^{\mathrm{GL}_n(\mathcal{O}_F)}(\tau_I)$$

We will show that irreducible sub-representations of $U_m(\tau_I)$ are atypical.

Theorem 3.0.9 (Main)**.** *Let $m \geq 1$. The $\mathrm{GL}_n(\mathcal{O}_F)$-irreducible subrepresentations of $U_m(\tau_I)$ are atypical.*

The classification of typical representations for the inertial class s is given by the following corollary.

Corollary 3.0.10. *The irreducible sub-representations of $\mathrm{ind}_{P_I(1)}^{\mathrm{GL}_n(\mathcal{O}_F)}(\tau_I)$ are precisely the typical representations for the level-zero inertial class $[M_I, \sigma_I]$. Moreover if Γ is a typical representation then*

$$\dim_{\mathbb{C}}\ \mathrm{Hom}_{\mathrm{GL}_n(\mathcal{O}_F)}(\Gamma, \mathrm{ind}_{P_I(1)}^{\mathrm{GL}_n(\mathcal{O}_F)}(\tau_I)) = \dim_{\mathbb{C}}\ \mathrm{Hom}_{\mathrm{GL}_n(\mathcal{O}_F)}(\Gamma, i_{P_I}^{\mathrm{GL}_n(F)}(\sigma_I)).$$

Proof. Given a typical representation Γ for the inertial class s, the theorem shows that Γ is a sub-representation of $\mathrm{ind}_{P_I(1)}^{\mathrm{GL}_n(\mathcal{O}_F)}(\tau_I)$ and the multiplicity formula follows from 2.2.4 and the above theorem. Conversely if Γ is a sub-representation of $\mathrm{ind}_{P_I(1)}^{\mathrm{GL}_n(\mathcal{O}_F)}(\tau_I)$ then by Frobenius reciprocity we get that $\mathrm{Hom}_{P_I(1)}(\tau_I, \Gamma) \neq 0$. If Γ is contained as a $\mathrm{GL}_n(\mathcal{O}_F)$-irreducible sub-representation in an irreducible smooth representation π of $\mathrm{GL}_n(F)$ then the restriction of π to $P_I(1)$ contains the representation τ_I. The pair $(P_I(1), \tau_I)$ is the Bushnell-Kutzko type for the inertial class $s = [M_I, \sigma_I]$ hence the inertial support of π is s. Hence Γ is a typical representation and this proves the corollary. $\square$

We will need a few lemmas before the proof of this theorem. Let $I = (n_1, n_2, \dots, n_r)$ be the ordered partition of the positive integer n as fixed at the beginning of this chapter. **Until the beginning of the section 3.1 we assume that** $r > 1$. We denote by I' the ordered partition $(n_1, n_2, \dots, n_{r-1})$ of $n - n_r$. Let m be a positive integer and $P_I(1, m)$ be the following set

$$\left\{ \begin{pmatrix} A & B \\ \varpi_F^m C & D \end{pmatrix} | A \in P_{I'}(1); B, C^t \in M_{n_r \times (n-n_r)}(\mathcal{O}_F); D \in \mathrm{GL}_{n_r}(\mathcal{O}_F) \right\}.$$

Note that $P_I(1,1) = P_I(1)$.

Lemma 3.0.11. *The set $P_I(1,m)$ is a subgroup of $P_I(1)$.*

Proof. The group $\mathrm{GL}_n(\mathcal{O}_F)$ acts on the set of lattices of F^n contained in the lattice ${\mathcal{O}_F}^n$. If $r-1=1$ the set $P_I(1,m)$ is the $\mathrm{GL}_n(\mathcal{O}_F)$-stabilizer of the lattice $(\mathcal{O}_F)^{n_1} \oplus (\varpi_F^m\,\mathcal{O}_F)^{n_2}$. In the case $r-1>1$ the set $P_I(1,m)$ is the $\mathrm{GL}_n(\mathcal{O}_F)$-stabilizer of lattices L_k for $1<k\leq r-1$ defined as:

$$L_k = (\mathcal{O}_F)^{n_1}\oplus\cdots\oplus(\mathcal{O}_F)^{n_{k-1}}\oplus(\varpi_F\,\mathcal{O}_F)^{n_k}\oplus\cdots\oplus(\varpi_F\,\mathcal{O}_F)^{n_{r-1}}\oplus(\varpi_F^m\,\mathcal{O}_F)^{n_r}.$$

This shows that $P_I(1,m)$ is a subgroup and is contained in $P_I(1)$ from the definition. $\square$

The structure of the representation

$$\mathrm{ind}_{P_I(1,m+1)}^{P_I(1,m)}(\mathrm{id})$$

will be used in the proof of the main theorem. Using Clifford theory we decompose the above representation. Let $K_I(m)$ be the group $K_n(m)U_{(n-n_r,n_r)}(\mathcal{O}_F)$.

Lemma 3.0.12. *The group $K_I(m)$ is a normal subgroup of $P_I(1,m)$ and $K_I(m)\cap P_I(1,m+1)$ is a normal subgroup of $K_I(m)$.*

Proof. The groups $K_I(m)$ and $P_I(1,m)$ satisfy Iwahori decomposition with respect to $U_{(n-n_r,n_r)}$, $\bar{U}_{(n-n_r,n_r)}$ and $M_{(n-n_r,n_r)}$. We also note that

$$K_I(m)\cap U_{(n-n_r,n_r)} = P_I(1,m)\cap U_{(n-n_r,n_r)}$$

and

$$K_I(m)\cap \bar{U}_{(n-n_r,n_r)} = P_I(1,m)\cap \bar{U}_{(n-n_r,n_r)}.$$

Hence $P_I(1,m)\cap U_{(n-n_r,n_r)}$ and $P_I(1,m)\cap\bar{U}_{(n-n_r,n_r)}$ normalize $K_I(m)$. Since $K_I(m)$ is a product of the group $K_n(m)$ and $U_{(n-n_r)}(\mathcal{O}_F)$ the group $P_I(1,m)\cap M_{(n-n_r,n_r)}$ normalizes the group $K_I(m)$. This shows the first part.

Notice that $K_I(m)\cap U_{(n-n_r,n_r)}$ is equal to $K_I(m)\cap P_I(1,m+1)\cap U_{(n-n_r,n_r)}$ and $K_I(m)\cap M_{(n-n_r,n_r)}$ is equal to $K_I(m)\cap P_I(1,m+1)\cap M_{(n-n_r,n_r)}$ hence it is enough to check that $K_I(m)\cap\bar{U}_{(n-n_r,n_r)}$ normalizes the group $K_I(m)\cap P_I(1,m+1)$. Since $K_I(m)\cap P_I(1,m+1)\cap\bar{U}_{(n-n_r,n_r)}$ is abelian and is contained in $K_I(m)\cap\bar{U}_{(n-n_r,n_r)}$ hence we need to check that $u^-j(u^-)^{-1}$ and $u^-u^+(u^-)^{-1}$ are contained in $K_I(m)\cap P_I(1,m+1)$ for all u^-, j and u^+ in

$K_I(m)\cap\bar{U}_{(n-n_r,n_r)}$,

$K_I(m)\cap P_I(1,m+1)\cap M_{(n-n_r,n_r)}$ and

$K_I(m)\cap P_I(1,m+1)\cap U_{(n-n_r,n_r)} = U_{(n-n_r,n_r)}(\mathcal{O}_F)$

respectively.

Let u^+, u^- and j be three elements from $U_{n-n_r,n_r}(\mathcal{O}_F)$, $K_I(m)\cap\bar{U}_{(n-n_r,n_r)}$ and $K_I(m)\cap P_I(1,m+1)\cap M_{(n-n_r,n_r)}$ respectively. We write them in their block form as:

$$u^+ = \begin{pmatrix} 1_{n-n_r} & B \\ 0 & 1_{n_r} \end{pmatrix}$$

where $B \in M_{(n-n_r)\times n_r}(\mathcal{O}_F)$,

$$u^- = \begin{pmatrix} 1_{n-n_r} & 0 \\ \varpi_F^m C & 1_{n_r} \end{pmatrix}$$

where $C \in M_{n_r\times(n-n_r)}(\mathcal{O}_F)$ and

$$j = \begin{pmatrix} J_1 & 0 \\ 0 & J_2 \end{pmatrix}.$$

We observe that $u^-j(u^-)^{-1} = j\{j^{-1}u^-j(u^-)^{-1}\}$ and the commutator $\{j^{-1}u^-j(u^-)^{-1}\}$ in its block form is as follows:

$$\begin{pmatrix} 1_{n-n_r} & 0 \\ J_2^{-1}(\varpi_F^m CJ_1^{-1} - \varpi_F^m C) & 1_{n_r} \end{pmatrix}.$$

We note that $J_2 \in K_{n_r}(m)$ and $J_1 \in K_{n-n_r}(m)$ hence $J_2^{-1}(\varpi_F^m CJ_1^{-1} - \varpi_F^m C)$ belongs to $\varpi_F^{m+1}M_{(n-n_r)\times n_r}(\mathcal{O}_F)$. This shows that

$$\{j^{-1}u^-j(u^-)^{-1}\} \in K_I(m)\cap P_I(m+1)$$

Now the element $(u^-)u^+(u^-)^{-1}$ is of the form

$$\begin{pmatrix} 1_{n-n_r} - \varpi_F^m BC & B \\ -\varpi_F^{2m} CBC & 1_{n_r} + \varpi_F^m CB \end{pmatrix}. \tag{3.1}$$

Since $2m \geq m+1$ the matrix in (3.1) is contained in the group $K_I(m)\cap P_I(1,m+1)$. $\square$

We now observe that $K_I(m)P_I(1,m+1) = P_I(1,m)$. From Mackey decomposition we get that

$$\mathrm{res}_{K_I(m)}\,\mathrm{ind}_{P_I(1,m+1)}^{P_I(1,m)}(\mathrm{id}) \simeq \mathrm{ind}_{K_I(m)\cap P_I(1,m+1)}^{K_I(m)}(\mathrm{id}).$$

Hence the above restriction decomposes into a direct sum of representations of the group

$$\frac{K_I(m)}{K_I(m)\cap P_I(1,m+1)}. \tag{3.2}$$

The inclusion map of $K_I(m)\cap\bar{U}_{(n-n_r,n_r)}$ in $K_I(m)$ induces the natural homomorphism

$$\tilde{\theta}_I:\ \frac{K_I(m)\cap\bar{U}_{(n-n_r,n_r)}}{P_I(1,m+1)\cap\bar{U}_{(n-n_r,n_r)}} \to \frac{K_I(m)}{K_I(m)\cap P_I(1,m+1)}.$$

Lemma 3.0.13. *The map $\tilde{\theta}_I$ is an $M_{(n-n_r,n_r)} \cap P_I(1,m)$ equivariant isomorphism.*

Proof. The map is clearly injective and surjectivity follows from the Iwahori decomposition of $K_I(m)$ with respect to the Levi-subgroup M_I. The inclusion of $K_I(m) \cap \bar{U}_{(n-n_r,n_r)}$ in $K_I(m)$ is an $M_{n-n_r,n_r} \cap P_I(1,m)$ equivariant map. $\square$

Let u^- be an element of the group $K_I(m) \cap \bar{U}_{(n-n_r,n_r)}$ and its block form be given by

$$\begin{pmatrix} 1_{(n-n_r,n_r)} & 0 \\ U^- & 1_{n_r} \end{pmatrix}.$$

The map $u^- \mapsto \varpi_F^{-m} U^-$ induces an isomorphism between the groups $K_I(m) \cap \bar{U}_{(n-n_r,n_r)}$ and $M_{n_r \times (n-n_r)}(\mathcal{O}_F)$. Let $\bar{U^-}$ be the image of U^- in the mod-$\mathfrak{P}_F$ reduction of $M_{n_r \times (n-n_r)}(\mathcal{O}_F)$. The map $u^- \mapsto \overline{\varpi_F^{-m} U^-}$ induces an isomorphism of the quotient (3.2) with the group of matrices $M_{n_r \times (n-n_r)}(k_F)$. The group $M_{(n-n_r,n_r)}(\mathcal{O}_F) = \mathrm{GL}_{n-n_r}(\mathcal{O}_F) \times \mathrm{GL}_{n_r}(\mathcal{O}_F)$ acts on the group $M_{n_r \times (n-n_r)}(k_F)$ through its mod-$\mathfrak{P}_F$ reduction $\mathrm{GL}_{n-n_r}(k_F) \times \mathrm{GL}_{n_r}(k_F)$, the action is given by $(g_1, g_2)U = g_2 U g_1^{-1}$ for all g_1 in $\mathrm{GL}_{n-n_r}(k_F)$, g_2 in $\mathrm{GL}_{n_r}(k_F)$ and U in $M_{n_r \times (n-n_r)}(k_F)$. The map $u^- \mapsto \overline{\varpi_F^{-m} U^-}$ is hence a $M_{(n-n_r,n_r)}(\mathcal{O}_F)$-equivariant map between the quotient (3.2) and $M_{n_r \times (n-n_r)}(k_F)$. Moreover the action of $M_{(n-n_r,n_r)}(\mathcal{O}_F)$ factors through its quotient $M_{(n-n_r,n_r}(k_F)$.

In general the group $G := \mathrm{GL}_m(\mathbb{F}_q) \times \mathrm{GL}_n(\mathbb{F}_q)$ acts on the set of matrices $M_{n\times m}(\mathbb{F}_q)$ by setting $(g_1, g_2)U = g_2 U g_1^{-1}$. We also have a G action on the set of matrices $M_{m\times n}(\mathbb{F}_q)$ by setting $(g_1, g_2)V = g_1 V g_2^{-1}$.

Lemma 3.0.14. *There exists a G-equivariant isomorphism between the groups $M_{m\times n}(\mathbb{F}_q)$ and $\widehat{M_{n\times m}(\mathbb{F}_q)}$.*

Proof. Let ψ be a non-trivial character of the additive group $\mathbb{F}_q$. We define a pairing B between $M_{m\times n}(\mathbb{F}_q)$ and $M_{n\times m}(\mathbb{F}_q)$ by defining $B(V,U) = \psi \circ \mathrm{tr}(VU)$. The pairing is non-degenerate and hence we obtain a map T between $M_{m\times n}(\mathbb{F}_q)$ and $\widehat{M_{n\times m}(\mathbb{F}_q)}$ defined by the equation

$$T(V)(U) = \psi \circ \mathrm{tr}(VU).$$

The map T is G equivariant since

$$(g_1, g_2)T(V)(U) = \psi \circ \mathrm{tr}(V g_2^{-1} U g_1) = \psi \circ \mathrm{tr}(g_1 V g_2^{-1} U) = T((g_1, g_2)V)(U).$$

$\square$

The above lemma gives a $M_{(n-n_r,n_r)} \cap P_I(1,m)$ equivariant map between the groups $\widehat{M_{n_r\times(n-n_r)}}(k_F)$ and $M_{(n-n_r)\times n_r}(k_F)$. Hence we get an

$$M_{(n-n_r,n_r)} \cap P_I(1,m)$$

equivariant isomorphism say θ_I between the group of characters of

$$\frac{K_I(m)}{K_I(m) \cap P_I(1, m+1)}$$

and the group of matrices $M_{(n-n_r)\times n_r}(k_F)$.

Since the group $K_I(m)$ is a normal subgroup of $P_I(1, m)$, we have an action of this group $P_I(1, m)$ on the set of characters of the abelian group

$$\frac{K_I(m)}{K_I(m) \cap P_I(1, m+1)}.$$

If η is one such character we denote by $Z(\eta)$ the $P_I(1, m)$-stabilizer of this character η. Clifford theory now gives the decomposition

$$\mathrm{ind}_{P_I(1,m+1)}^{P_I(1,m)}(\mathrm{id}) \simeq \bigoplus_{\eta_k} \mathrm{ind}_{Z(\eta_k)}^{P_I(1,m)}(U_{\eta_k})$$

where $\{\eta_k\}$ is a set of representatives for the orbits under the action of $P_I(1, m)$ and U_{η_k} is some irreducible representation of the group $Z(\eta_k)$. We also note that $Z(\mathrm{id}) = P_I(1, m)$ and the identity character occurs with multiplicity one (which follows from Frobenius reciprocity) and hence

$$\mathrm{ind}_{P_I(1,m+1)}^{P_I(1,m)}(\mathrm{id}) \simeq \mathrm{id} \oplus \bigoplus_{\eta_k \neq \mathrm{id}} \mathrm{ind}_{Z(\eta_k)}^{P_I(1,m)}(U_{\eta_k}). \tag{3.3}$$

Observe that

$$Z(\eta_k) = (Z(\eta_k) \cap M_{(n-n_r,n_r)})K_I(m).$$

Since we have a $M_{(n-n_r,n_r)} \cap P_I(1, m)$ equivariant map between the group of characters of (3.2) and $M_{(n-n_r)\times n_r}(k_F)$, note that

$$Z(\eta_k) \cap M_{(n-n_r,n_r)} = Z_{M_{(n-n_r,n_r)}\cap P_I(1,m)}(A)$$

for some matrix A in $M_{(n-n_r)\times n_r}(k_F)$. The group $M_{(n-n_r,n_r)} \cap P_I(1, m)$ acts on the group of matrices $M_{(n-n_r,n_r)}(k_F)$ through its mod-$\mathfrak{P}_F$ reduction. The mod-$\mathfrak{P}_F$ reduction of the group $P_I(1, m) \cap M_{(n-n_r,n_r)}$ is equal to the group $P_{I'}(k_F) \times \mathrm{GL}_{n_r}(k_F)$. In the next lemma we will bound the mod $\mathfrak{P}_F$ reduction of the group $Z(\eta_k) \cap M_I$ for the proof of the main theorem . Let $\mathcal{O}_A$ be an orbit for the action of $P_{I'}(k_F) \times \mathrm{GL}_{n_r}(k_F)$ on the set of matrices $M_{(n-n_r)\times n_r}(k_F)$. Let p_j be the j^{th} projection of the group $M_I(k_F) = \prod_{i=1}^{r} \mathrm{GL}_{n_i}(k_F)$.

Lemma 3.0.15. *Let $\mathcal{O}_A$ be an orbit consisting of non-zero matrices in*

$$M_{(n-n_r)\times n_r}(k_F).$$

We can choose a representative A such that the $P_{I'}(k_F) \times \mathrm{GL}_{n_r}(k_F)$-stabilizer of A,

$$Z_{P_{I'}(k_F)\times \mathrm{GL}_{n_r}(k_F)}(A)$$

satisfies one of the following conditions.

1. *There exists a positive integer j, $j \leq r$ such that the image of*

$$p_j : Z_{P_{I'}(k_F)\times \mathrm{GL}_{n_r}(k_F)}(A) \cap M_I(k_F) \to \mathrm{GL}_{n_j}(k_F)$$

is contained in a proper parabolic subgroup of $\mathrm{GL}_{n_j}(k_F)$.

2. *There exists an i with $1 \leq i \leq r-1$ such that $p_i(g) = p_r(g)$ for all $g \in Z_{P_{I'}(k_F)\times \mathrm{GL}_{n_r}(k_F)}(A) \cap M_I(k_F)$.*

Proof. Let $A = [U_1, U_2, \ldots, U_{(r-1)}]^{tr}$ be the block form (U_k is a matrix of size $n_r \times n_k$ for $1 \leq k \leq r-1$) of a representative m for an orbit $\mathcal{O}_m$ consisting of non-zero matrices. If $((A_{ij}), B) \in Z_{P_{I'}(k_F)\times \mathrm{GL}_{n_r}(k_F)}(A)$ then we have

$$(A_{ij})[U_1, U_2, ..., U_{(r-1)}]^{tr} = [U_1, U_2, ..., U_{(r-1)}]^{tr} B. \tag{3.4}$$

Since $(A_{ij}) \in P_{I'}(k_F)$, we have $A_{ij} = 0$ for all $i > j$. Let $l' \leq r-1$ be the least non-negative integer such that $U_{r-1-l'}$ (A_{ii} matrix of size $n_i \times n_i$) is non-zero and such an l' exists since $m \neq 0$. From (3.4) we get that $A_{ll}U_l^{tr}t = U_l^{tr}B$ where $l = r-1-l'$. There exist matrices $P \in \mathrm{GL}_{n_r}(k_F)$ and $Q \in \mathrm{GL}_{n_l}(k_F)$ such that PU_l^tQ is a matrix of the form

$$\begin{pmatrix} 1_t & 0 \\ 0 & 0 \end{pmatrix} \tag{3.5}$$

where t is the rank of the matrix U_l^{tr}. Now we may change the representative A to $A' = [U_1', U_2', \ldots, U_r']^{tr}$ by the action of the element

$$\mathrm{diag}(1_{n_1}, \ldots, P, \ldots, 1_{n_{r-1}}, Q^{-1})$$

in $P_{I'}(k_F) \times \mathrm{GL}_{n_r}(k_F)$ such that $U_l'^{tr}$ is the matrix (3.5). If $t = n_l = n_r$ then condition (2) is satisfied. Consider the maps $T_1 : k_F^{n_l} \to k_F^{n_r}$ and $T_2 : k_F^{n_r} \to k_F^{n_l}$ given by

$$(a_1, a_2, \ldots, a_{n_l}) \mapsto (a_1, a_2, \ldots, a_{n_l})U_l^{tr}$$

and

$$(a_1, a_2, \ldots, a_{n_r}) \mapsto U_l^{tr}(a_1, a_2, \ldots, a_{n_r})^{tr}$$

respectively. If $t = n_l = n_r$ does not hold then either of T_1 or T_2 has a non-trivial proper kernel (since $U_l \neq 0$). If T_1 has a non-trivial proper kernel then A_{ll} preserves this kernel and hence belongs to a proper parabolic subgroup of $\mathrm{GL}_{n_r}(k_F)$. If T_2 has a non-trivial proper kernel then B preserves this kernel and hence belongs to a proper parabolic subgroup of $\mathrm{GL}_{n_l}(k_F)$. Hence if $t = n_l = n_r$ does not hold true then condition (1) is satisfied. □

The following lemma is due to Paskunas but we give a mild modification for our applications (see [Pas05, Proposition 6.8]).

Lemma 3.0.16. *Let $m > 1$, σ be any irreducible representation of the group $\mathrm{GL}_m(\mathbb{F}_q)$ and H be a subgroup contained in a proper parabolic subgroup of $\mathrm{GL}_m(\mathbb{F}_q)$. For every irreducible representation γ of H contained in $\mathrm{res}_H(\sigma)$, there exists an irreducible non-cuspidal representation σ' of $\mathrm{GL}_m(\mathbb{F}_q)$ such that $\mathrm{Hom}_H(\gamma, \sigma') \neq 0$.*

Proof. Let P be a proper parabolic subgroup of $\mathrm{GL}_m(\mathbb{F}_q)$ containing H and $\bar{U}$ be the unipotent radical of an opposite parabolic subgroup of P. We observe that $\bar{U} \cap H = \mathrm{id}$. Now if the lemma is false, we have $\mathrm{ind}_H^{\mathrm{GL}_m(\mathbb{F}_q)}(\gamma) \simeq \oplus_{k\in\Lambda}\sigma_k$ such that σ_k is a cuspidal representation. Using Mackey decomposition we get that

$$\mathrm{Hom}_{\bar{U}}(\mathrm{id}, \mathrm{ind}_H^{\mathrm{GL}_m(\mathbb{F}_q)}(\gamma)) = \bigoplus_{u\in\bar{U}\backslash \mathrm{GL}_m(\mathbb{F}_q)/H} \mathrm{Hom}_{\bar{U}\cap H^u}(\mathrm{id}, \gamma^u).$$

If $\bar{U} \cap H = \mathrm{id}$ then $\mathrm{Hom}_{H\cap\bar{U}}(\mathrm{id}, \gamma)$ is non-zero and by the above decomposition

$$\mathrm{Hom}_{\bar{U}}(\mathrm{id}, \mathrm{ind}_H^{\mathrm{GL}_m(\mathbb{F}_q)}(\gamma)) \neq 0.$$

This shows that $\mathrm{Hom}_{\bar{U}}(\mathrm{id}, \sigma_k) \neq 0$ for some $k \in \Lambda$ and this is a contradiction to our assumption. $\square$

Lemma 3.0.17. *Let $m \geq 2$, H be the diagonal subgroup of $\mathrm{GL}_m(\mathbb{F}_q) \times \mathrm{GL}_m(\mathbb{F}_q)$ and $\sigma_1 \boxtimes \sigma_2$ be an irreducible representation of $\mathrm{GL}_m(\mathbb{F}_q) \times \mathrm{GL}_m(\mathbb{F}_q)$. For every irreducible representation γ occurring in $\mathrm{res}_H\, \sigma_1 \boxtimes \sigma_2$ there exists an irreducible non-cuspidal representation $\sigma_1' \boxtimes \sigma_2'$ of $\mathrm{GL}_m(\mathbb{F}_q) \times \mathrm{GL}_m(\mathbb{F}_q)$ containing γ.*

Proof. Let $\bar{U}$ and U be the subgroups of lower unipotent and upper unipotent matrices of $\mathrm{GL}_m(\mathbb{F}_q)$. Consider the unipotent subgroup $V := \bar{U} \times U$ of $\mathrm{GL}_m(\mathbb{F}_q) \times \mathrm{GL}_m(\mathbb{F}_q)$. Suppose the lemma is false then

$$\mathrm{ind}_H^{\mathrm{GL}_m(\mathbb{F}_q)\times\mathrm{GL}_m(\mathbb{F}_q)}(\gamma) \simeq \oplus_{k\in\Lambda}\sigma_1^k \boxtimes \sigma_2^k$$

such that σ_1^k and σ_2^k are cuspidal representations for all $k \in \Lambda$. We observe that $V \cap H = \mathrm{id}$ and by Mackey decomposition we have

$$\mathrm{Hom}_V(\mathrm{id}, \mathrm{ind}_H^{\mathrm{GL}_m(\mathbb{F}_q)\times\mathrm{GL}_m(\mathbb{F}_q)}(\gamma)) \neq 0.$$

Now by our assumption we have $\mathrm{Hom}_V(\mathrm{id}, \sigma_1^k \boxtimes \sigma_2^k) \neq 0$ for some $k \in \Lambda$ and hence a contradiction. $\square$

The following lemma is similar to proposition 2.2.3. The lemma is just a modified version of the proposition 2.2.3 for our present use.

Lemma 3.0.18. *Let Γ be a $\mathrm{GL}_{n-n_r}(\mathcal{O}_F)$-irreducible sub-representation of*

$$\mathrm{ind}_{P_{(n-n_r,n_r)}(m)}^{\mathrm{GL}_n(\mathcal{O}_F)}\{U_m(\tau_{I'}) \boxtimes \tau_r\}.$$

If the irreducible sub-representations of $U_m(\tau_{I'})$ are atypical for the component $s = [M_{I'}, \sigma_{I'}]$, then the representation Γ is atypical for the component $s = [M_I, \sigma_I]$.

Proof. Let ρ be an irreducible sub-representation of $U_m(\tau_{I'})$. If ρ is not typical then, there exists another Bernstein component $[M_J, \lambda_J]$ of $\mathrm{GL}_{n-n_r}(F)$ such that

$$[M_{I'}, \sigma_{I'}] \neq [M_J, \lambda_J]$$

and ρ is contained in

$$\mathrm{res}_{\mathrm{GL}_{n-n_r}(\mathcal{O}_F)}\, i_{P_J}^{\mathrm{GL}_{(n-n_r)}(F)}(\lambda_J)$$

where $J = (n'_1, n'_2, \ldots, n'_{r'-1})$ and $\lambda_J = \boxtimes_{i=1}^{r'-1} \lambda_i$. The representation

$$\mathrm{ind}_{P_{(n-n_r,n_r)}(m)}^{\mathrm{GL}_n(\mathcal{O}_F)} \{\rho \boxtimes \tau_r\}$$

is contained in

$$\mathrm{ind}_{P_{(n-n_r,n_r)} \cap \mathrm{GL}_n(\mathcal{O}_F)}^{\mathrm{GL}_n(\mathcal{O}_F)} \{\rho \boxtimes \tau_r\}. \tag{3.6}$$

The representation (3.6) is contained in the representation

$$\mathrm{res}_{\mathrm{GL}_n(\mathcal{O}_F)}\, i_{P_{(n-n_r,n_r)}}^{\mathrm{GL}_n(F)} \{i_{P_J}^{\mathrm{GL}_{n-n_r}(F)}(\lambda_J) \boxtimes \sigma_r\}.$$

Since $[M_{I'}, \sigma_{I'}] \neq [M_J, \lambda_J]$ there exist an inertial class $[\mathrm{GL}_p(F), \sigma]$ occurring in the multi-set

$$\{[\mathrm{GL}_{n_1}(F), \sigma_1], [\mathrm{GL}_{n_2}(F), \sigma_2], \ldots, [\mathrm{GL}_{n_{r-1}}(F), \sigma_{r-1}]\}$$

with a multiplicity not equal to its multiplicity in the multi-set

$$\{[\mathrm{GL}_{n'_1}(F), \lambda_1], [\mathrm{GL}_{n'_2}(F), \lambda_2], \ldots, [\mathrm{GL}_{n'_{r'-1}}(F), \lambda_{r'-1}]\}.$$

Hence the classes $[M_I, \sigma_I]$ and $[M_J \times \mathrm{GL}_{n_r}(F), \lambda_J \boxtimes \sigma_r]$ represent two distinct Bernstein components for the group $\mathrm{GL}_n(F)$. □

3.1 Proof of the main theorem

Proof of theorem 3.0.9. We prove the theorem by using induction on the positive integer n, the rank of $\mathrm{GL}_n(F)$. The theorem is true for $n = 1$ since $U_m(\tau_I)$ is zero. We assume that the theorem is true for all positive integers less than $n+1$. We will show the theorem for the positive integer $n+1$. Let $s = [M_I, \sigma_I]$ be a level-zero inertial class. We assume that the partition $I = (n_1, n_2, \ldots, n_r)$ of $n+1$ satisfies the hypothesis $n_i \leq n_j$ for all $1 \leq i \leq j \leq r$. If $r = 1$ we have $U_m(\tau_I) = 0$ and the theorem holds by default. We now assume that $r > 1$ and let $I' = (n_1, n_2, \ldots, n_{r-1})$.

We now break the proof into two cases. The first case is $n_r = 1$ and the second case is $n_r > 1$.

3.1.1 The case where $n_r = 1$

In this case $n_i = 1$ for $1 \leq i \leq r$ and $P_I = B_n$ where B_n is the Borel subgroup of GL_n. We denote by T_n and U_n the maximal torus and the unipotent radical respectively. We also use the notation $B_n(m)$ for the subgroup $P_I(m)$ and χ_{I_n} for τ_I since $I = (1, 1, \ldots, 1)$ is a tuple of length n. The proof is by induction on the integer n, the rank of T_n. The statement is immediate for $n = 1$ and for $n = 2$ we refer to [BM02, A.2.4] for a proof (We will require the proof for later use and we will recall it at that stage). So we prove the theorem for $n \geq 3$. Suppose the theorem is true for some positive integer $n \geq 2$. The rest of this subsection is to prove the main theorem for $n+1$. By definition of $U_m(\chi_{I_{n+1}})$ we have

$$\mathrm{ind}_{B_{n+1}(m)}^{\mathrm{GL}_{n+1}(\mathcal{O}_F)}(\chi_{I_{n+1}}) \simeq U_m(\chi_{I_{n+1}}) \oplus \mathrm{ind}_{B_{n+1}(1)}^{\mathrm{GL}_{n+1}(\mathcal{O}_F)}(\chi_{I_{n+1}}).$$

We have the isomorphism

$$\mathrm{ind}_{B_{n+1}(m)}^{\mathrm{GL}_{n+1}(\mathcal{O}_F)}(\chi_{I_{n+1}}) \simeq \mathrm{ind}_{P_{(n,1)}(m)}^{\mathrm{GL}_{n+1}(\mathcal{O}_F)}\{\mathrm{ind}_{B_n(m)}^{\mathrm{GL}_n(\mathcal{O}_F)}(\chi_{I_n}) \boxtimes \chi_{n+1})\}.$$

We also have the decomposition

$$\begin{aligned}\mathrm{ind}_{P_{(n,1)}(m)}^{\mathrm{GL}_{n+1}(\mathcal{O}_F)}\{\mathrm{ind}_{B_n(m)}^{\mathrm{GL}_n(\mathcal{O}_F)}(\chi_{I_n}) \boxtimes \chi_{n+1}\} \simeq \\ \mathrm{ind}_{P_{(n,1)}(m)}^{\mathrm{GL}_{n+1}(\mathcal{O}_F)}\{U_m(\chi_{I_n}) \boxtimes \chi_n\} \oplus \mathrm{ind}_{P_{(n,1)}(m)}^{\mathrm{GL}_{n+1}(\mathcal{O}_F)}\{\mathrm{ind}_{B_n(1)}^{\mathrm{GL}_n(\mathcal{O}_F)}(\chi_{I_n}) \boxtimes \chi_{n+1}\}.\end{aligned}$$

By induction hypothesis and lemma 3.0.18 irreducible sub-representations of

$$\mathrm{ind}_{P_{(n,1)}(m)}^{\mathrm{GL}_{n+1}(\mathcal{O}_F)}\{U_m(\chi_{I_n}) \boxtimes \chi_{n+1}\}$$

are atypical representations. We now consider the irreducible factors of the representation

$$\mathrm{ind}_{P_{(n,1)}(m)}^{\mathrm{GL}_{n+1}(\mathcal{O}_F)}\{\mathrm{ind}_{B_n(1)}^{\mathrm{GL}_n(\mathcal{O}_F)}(\chi_{I_n}) \boxtimes \chi_{n+1}\}. \tag{3.7}$$

We use induction on the integer m to show that the representation

$$\begin{aligned}&\mathrm{ind}_{P_{(n,1)}(1)}^{\mathrm{GL}_{n+1}(\mathcal{O}_F)}\{\mathrm{ind}_{B_n(1)}^{\mathrm{GL}_n(\mathcal{O}_F)}(\chi_{I_n}) \boxtimes \chi_{n+1}\} \\ &\simeq \mathrm{ind}_{B_{n+1}(1)}^{\mathrm{GL}_{n+1}(\mathcal{O}_F)}(\chi_{I_{n+1}})\end{aligned}$$

has a complement say $U_{1,m}(\chi_{In+1})$ in the representation (3.7) whose irreducible sub-representations are all atypical representations. This shows that irreducible sub-representations of $U_m(\chi_{I_{n+1}})$ are atypical. To reduce the notations we denote by $P(m)$ the subgroup $P_{(n,1)}(m)$. Applying the decomposition (3.3) to the parabolic subgroup $P_{(n,1)}$ we get that

$$\mathrm{ind}_{P(m+1)}^{P(m)}(\mathrm{id}) = \mathrm{id} \oplus \mathrm{ind}_{Z(\eta)}^{P(m)}(U_\eta)$$

where η (in the present situation we just have one orbit consisting of non-trivial characters) is any non-trivial character of the group $K_{n+1}(m)U_{n,1}(\mathcal{O}_F)$ which is trivial on
$K_{n+1}(m)U_{n,1}(\mathcal{O}_F)\cap P(m+1)$ and $K_{n+1}(m)$ is the principal congruence subgroup of level m. We have shown a $M\cap P(m)$ equivariant map between the group of characters of

$$\frac{K_{n+1}(m)U_{n,1}(\mathcal{O}_F)}{K_{n+1}(m)U_{n,1}(\mathcal{O}_F)\cap P(m+1)}$$

and $M_{n\times 1}(k_F)$. We choose η to be the character corresponding to the matrix $[1,0\ldots,0]$.

For the above choice of a non-trivial character we have

$$\begin{aligned}
&\mathrm{ind}_{P(m+1)}^{\mathrm{GL}_{n+1}(\mathcal{O}_F)}\{\mathrm{ind}_{B_n(1)}^{\mathrm{GL}_n(\mathcal{O}_F)}(\chi_{I_n})\boxtimes\chi_{n+1}\}\\
&\simeq \mathrm{ind}_{P(m)}^{\mathrm{GL}_{n+1}(\mathcal{O}_F)}\{\mathrm{ind}_{B_n(1)}^{\mathrm{GL}_n(\mathcal{O}_F)}(\chi_{I_n})\boxtimes\chi_{n+1}\}\\
&\qquad\oplus \mathrm{ind}_{Z(\eta)}^{\mathrm{GL}_{n+1}(\mathcal{O}_F)}\{U_\eta\otimes \mathrm{res}_{Z(\eta)\cap M_{(n,1)}}\{\mathrm{ind}_{B_n(1)}^{\mathrm{GL}_n(\mathcal{O}_F)}(\chi_{I_n})\boxtimes\chi_{n+1}\}\}.
\end{aligned}$$

Since the representation $\mathrm{ind}_{B_n(1)}^{\mathrm{GL}_n(\mathcal{O}_F)}(\chi_{I_n})\boxtimes\chi_{n+1}$ is a level-zero representation,

$$\mathrm{res}_{Z(\eta)\cap M_{(n,1)}}\{\mathrm{ind}_{B_n(1)}^{\mathrm{GL}_n(\mathcal{O}_F)}(\chi_{I_n})\boxtimes\chi_{n+1}\}$$

is isomorphic to the inflation of the representation

$$\mathrm{res}_{\overline{Z(\eta)\cap M_{(n,1)}}}\{\mathrm{ind}_{B_n(k_F)}^{\mathrm{GL}_n(k_F)}(\chi_{I_n})\boxtimes\chi_{n+1}\}$$

where $\overline{Z(\eta)\cap M_{(n,1)}}$ is the mod-$\mathfrak{P}_F$ reduction of the group $Z(\eta)\cap M_{(n,1)}$. The group $\overline{Z(\eta)\cap M_{(n,1)}}$ is contained in the following subgroup

$$\left\{\begin{pmatrix}A & B & 0\\ 0 & d & 0\\ 0 & 0 & d\end{pmatrix}|A\in\mathrm{GL}_{n-1}(k_F), B\in M_{(n-1)\times 1}(k_F) \text{ and } d\in k_F^\times\right\}. \tag{3.8}$$

Let Mir_k be the following group

$$\left\{\begin{pmatrix}A & B\\ 0 & 1\end{pmatrix}|A\in\mathrm{GL}_{k-1}(k_F), B\in M_{(k-1)\times 1}(k_F),\right\}$$

Now we have to understand the restriction

$$\mathrm{res}_{P_{(n-1,1)}}\mathrm{ind}_{B_n(k_F)}^{\mathrm{GL}_n(k_F)}(\chi_{I_n})$$

which is reduced to understanding the restriction

$$\mathrm{res}_{\mathrm{Mir}_{n-1}}\mathrm{ind}_{B_n(k_F)}^{\mathrm{GL}_n(k_F)}(\chi_{I_n}).$$

We use the theory of derivatives (originally for $\mathrm{GL}_n(F)$ due to Bernstein and Zelevinsky (see [BZ76])) to describe this restriction in a way sufficient for our application. We refer to [Zel81, Chapter 3, §13] for details of these constructions.

In the case of finite fields from Clifford theory one can define four exact functors and we recall the formalism here. The precise definitions are not required for our purpose except for one functor Ψ^+ which will be recalled latter:

$$\mathcal{M}(\mathrm{Mir}_{k-1}) \underset{\Phi^-}{\overset{\Phi^+}{\rightleftarrows}} \mathcal{M}(\mathrm{Mir}_k) \underset{\Psi^+}{\overset{\Psi^-}{\rightleftarrows}} \mathcal{M}(\mathrm{GL}_{k-1}(k_F))$$

The key results we use from Zelevinsky are summarised below (see [Zel81, Chapter 3, §13]).

Theorem 3.1.1 (Zelevinsky)**.** *The functors Ψ^+ and Φ^- are left adjoint to Ψ^- and Φ^+ respectively. The compositions $\Phi^-\Phi^+$ and $\Psi^-\Psi^+$ are naturally equivalent to identity. Moreover $\Phi^+\Psi^-$ and $\Phi^-\Psi^+$ are zero. The diagram*

$$0 \to \Phi^+\Phi^- \to \mathrm{id} \to \Psi^+\Psi^- \to 0$$

obtained from these properties is exact.

Using this theorem and following Bernstein-Zelevinsky one can define a filtration Fil on a finite dimensional representation τ of Mir_n, for all $n > 1$. The filtration Fil is given by

$$0 \subset \tau_n \subset \ldots \subset \tau_3 \subset \tau_2 \subset \tau_1 = \tau$$

where $\tau_k = (\Phi^+)^{k-1}(\Phi^-)^{k-1}$ and $\tau_k/\tau_{k+1} = (\Phi^+)^{k-1}\Psi^+\Psi^-(\Phi^-)^{k-1}(\tau)$ for all $k \geq 1$. The representation $\tau^{(k)} := \Psi^-(\Phi^-)^{k-1}(\tau)$ for all $k \geq 0$ of $\mathrm{GL}_{n-k}(k_F)$ is called the k^{th}-derivative of τ and by convention $\tau^{(0)} := \tau$.

Let R_n be the Grothendieck group of $\mathrm{GL}_n(k_F)$ for all $n \geq 1$ and set $R_0 = \mathbb{Z}$. Zelevinsky defined a ring structure on the group $R = \oplus_{n\geq 0} R_n$ by setting parabolic induction as the product rule. Recall that the ring R has a $\mathbb{Z}$-linear map D defined by setting $D(\pi) = \sum_{k\geq 0}(\pi|_{\mathrm{Mir}_n})^{(k)}$ for all π in R_n. It follows from [Zel81, Chapter 3, §13] that

$$D(\mathrm{ind}_P^{\mathrm{GL}_n(k_F)}(\tau_1 \boxtimes \cdots \boxtimes \tau_r)) = \prod_{i=1}^{r} D(\tau_i)$$

where the product on the right hand side is in the ring R. The map D is hence an endomorphism of the ring R. If π is a supercuspidal representation of $\mathrm{GL}_n(k_F)$ then by Gelfand-Kazhdan theory it follows that $\pi^{(n)} = 1$, $\pi^{(0)} = \pi$

and all other derivatives are zero (see [Zel81, Chapter 3, §13]). Let $1_R \in R_0$ be the identity element of R.

In our present situation we have

$$D(\operatorname{ind}_{B_n(1)}^{\mathrm{GL}_n(\mathcal{O}_F)}(\chi_{I_n})) = \prod_{i=1}^{n} D(\chi_i) = \prod_{i=1}^{n}(\chi_i + 1_R).$$

Let X_{n-k} be the term of degree $(n-k)$ in the expansion of the above product (it is a representation of $\mathrm{GL}_{n-k}(k_F)$ in the Grothendieck group R_{n-k}. Since the coefficients of the above expansion are positive X_{n-k} is actually a representation and not just a virtual representation.) Then we have

$$\operatorname{res}_{\mathrm{Mir}_{n-1}} \operatorname{ind}_{B_n(k_F)}^{\mathrm{GL}_n(k_F)}(\chi_{I_n}) \simeq \bigoplus_{k\geq 1}^{n} (\Phi^+)^{k-1}\Psi^+(X_{n-k}).$$

Observe that $P_{(n-1,1)} = \mathrm{Mir}_{(n-1)}\, k_F^\times$ (here $k_F^\times$ is the centre of $\mathrm{GL}_n(k_F)$) and $\mathrm{Mir}_{(n-1)} \cap k_F^\times = \mathrm{id}$. The representation

$$\rho := (\Phi^+)^{k-1}\Psi^+(X_{n-k})$$

extends to a representation of $P_{(n-1,1)}$ by setting $\rho(a) = \chi(a)$ for all $a \in k_F^\times$ where χ is the central character of the representation

$$\operatorname{ind}_{B_n(k_F)}^{\mathrm{GL}_n(k_F)}(\boxtimes_{i=1}^{n}\chi_i).$$

Since the central character will play some role, we denote the extended representation by

$$ext\{(\Phi^+)^{k-1}\Psi^+(X_{n-k})\}.$$

By inflation we extend the $P_{(n,1)}(k_F) \times k_F^\times$-representation

$$ext\{(\Phi^+)^{k-1}\Psi^+(X_{n-k})\} \boxtimes \chi_{n+1}$$

to a representation of $Z(\eta) \cap M_{(n,1)}$. We continue to use the notation

$$ext\{(\Phi^+)^{k-1}\Psi^+(X_{n-k})\} \boxtimes \chi_{n+1}$$

for the extended representation. We now have

$$\begin{aligned}\operatorname{ind}_{P(m+1)}^{\mathrm{GL}_{n+1}(\mathcal{O}_F)}(\chi_{I_n}) \simeq{}& \operatorname{ind}_{P(m)}^{\mathrm{GL}_{n+1}(\mathcal{O}_F)}(\chi_{I_n}) \oplus \\ & \bigoplus_{k\geq 1}^{n} \operatorname{ind}_{Z(\eta)}^{\mathrm{GL}_{n+1}(\mathcal{O}_F)}\{ext\{(\Phi^+)^{k-1}\Psi^+(X_{n-k})\} \boxtimes \chi_{n+1}\}.\end{aligned}$$

We will show that any irreducible sub-representation of

$$\operatorname{ind}_{Z(\eta)}^{\mathrm{GL}_{n+1}(\mathcal{O}_F)}(ext\{(\Phi^+)^{k-1}\Psi^+(X_{n-k})\} \boxtimes \chi_{n+1})\}$$

is atypical for the component $[T_n, \chi_{I_n}]$.

We first consider the case when $k \geq 2$. The representation X_{n-k} is a direct sum of the representations:

$$\mathrm{ind}_{B_{n-k}(k_F)}^{\mathrm{GL}_{n-k}(k_F)}(\chi_{i_1} \boxtimes \chi_{i_2} \boxtimes \ldots \cdots \boxtimes \chi_{i_{n-k}}).$$

The above term also occurs in the expansion

$$\prod_{j=1}^{n-k}(1_R + \chi_{i_j})(1_R + \lambda)$$

where λ is a cuspidal representation of $\mathrm{GL}_k(k_F)$. To shorten the notation we use the symbol $\times$ for the multiplication in the ring R. We get that the representation

$$(\Phi^+)^{k-1}\Psi^+(\times_{j=1}^{n-k}\chi_{n_j})$$

occurs in the representation

$$\mathrm{res}_{\mathrm{Mir}_{n-1}}(\times_{j=1}^{n-k}\chi_{n_j} \times \lambda).$$

Since the mod-$\mathfrak{P}_F$ reduction of the group $Z(\eta) \cap M_{(n,1)}$ is contained in the subgroup of the form (3.8), even if the central characters of $\times_{j=1}^{n}\chi_j$ and $\times_{j=1}^{n-k}(\chi_j) \times \lambda$ are different we may change χ_{n+1} to χ'_{n+1} such that the representation

$$\mathrm{res}_{Z(\eta)\cap M_{(n,1)}}\{ext((\Phi^+)^{k-1}\Psi^+(\times_{j=1}^{n-k}\chi_{n_j}))\} \boxtimes \chi_{n+1}$$

occurs in the representation

$$\mathrm{res}_{Z(\eta)\cap M_{(n,1)}}(\times_{j=1}^{n-k}(\chi_j) \times \lambda) \boxtimes \chi'_{n+1}.$$

Hence an irreducible sub-representation of

$$\mathrm{ind}_{Z(\eta)}^{\mathrm{GL}_{n+1}(\mathcal{O}_F)}\{(ext\{(\Phi^+)^{k-1}\Psi^+(X_{n-k})\} \boxtimes \chi_{n+1}) \otimes U_\eta\} \tag{3.9}$$

occurs as a sub-representation of

$$\mathrm{ind}_{Z(\eta)}^{\mathrm{GL}_{n+1}(\mathcal{O}_F)}\{\{(\chi_{n_1} \boxtimes \chi_{n_2} \boxtimes \cdots \boxtimes \chi_{n_k} \boxtimes \lambda \boxtimes \chi'_{n+1})\} \otimes U_\eta\}. \tag{3.10}$$

The above representation occurs as a sub-representation of

$$\mathrm{ind}_{P_{(1,1,\ldots,k,1)}\cap\mathrm{GL}_{n+1}(\mathcal{O}_F)}^{\mathrm{GL}_{n+1}(\mathcal{O}_F)}\{\chi_{n_1} \boxtimes \chi_{n_2} \boxtimes \cdots \boxtimes \chi_{n_k} \boxtimes \lambda \boxtimes \chi'_{n+1}\}. \tag{3.11}$$

Hence the sub-representation of (3.9) are not typical representations.

Now we are left with the term

$$\mathrm{ind}_{Z(\eta)}^{\mathrm{GL}_{n+1}(\mathcal{O}_F)}\{(ext\{\Psi^+(X_{n-1})\} \boxtimes \chi_{n+1}) \otimes U_\eta\}. \tag{3.12}$$

We might as well repeat the same strategy as for $k \geq 2$ and now λ is one dimensional but the representations (3.11) and $\times_{j=1}^{n+1}\chi_j$ may not have distinct inertial support. In order to tackle the terms of the above representation we use a different technique. We now recall the definition of the representation U_η, the functor Ψ^+ and some facts due to Casselman regarding the restriction of an irreducible smooth representation to the maximal compact subgroup $\mathrm{GL}_2(\mathcal{O}_F)$.

The representation U_η is a character on the group $Z(\eta)$. From (3.8) any element of the group $Z(\eta)$ is of the form

$$\begin{pmatrix} A & B & X' \\ \varpi_F C & d & y \\ \varpi_F^m X & \varpi_F^m y' & e \end{pmatrix} \tag{3.13}$$

where $A \in \mathrm{GL}_{n-1}(\mathcal{O}_F)$; $(X')^t, X^t, B, C^t \in M_{(n-1)\times 1}(\mathcal{O}_F)$; $e, d \in \mathcal{O}_F^\times$; $y, y' \in \mathcal{O}_F$ and $d \equiv e(\mathfrak{P}_F)$. The character U_η is given by

$$\begin{pmatrix} A & B & X' \\ \varpi_F C & d & y \\ \varpi_F^m X & \varpi_F^m y' & e \end{pmatrix} \mapsto \eta(\varpi_F^m y).$$

The functor

$$\Psi^+ : \mathcal{M}(\mathrm{GL}_{k-1}(k_F)) \to \mathcal{M}(\mathrm{Mir}_k)$$

is the inflation functor via the quotient map of Mir_k modulo the unipotent radical of Mir_k.

Let (π, V_π) be an irreducible smooth representation of $\mathrm{GL}_2(F)$. We denote by $c(\pi)$ and ϖ_π the conductor and central character of the representation π respectively. Let V^N be the space of all vectors fixed by the principal congruence subgroup of level N for all $N \geq 1$. For all $i > c(\varpi_\pi)$ we define the representation $U_i(\chi)$ as the complement of the representation $\mathrm{ind}_{B_2(i-1)}^{\mathrm{GL}_2(\mathcal{O}_F)}(\chi)$ in $\mathrm{ind}_{B_2(i)}^{\mathrm{GL}_2(\mathcal{O}_F)}(\chi)$. For $i = c(\varpi_\pi)$ we set

$$U_i(\varpi_\pi) = \mathrm{ind}_{B_2(i)}^{\mathrm{GL}_2(\mathcal{O}_F)}(\varpi_\pi \boxtimes \mathrm{id}).$$

It follows from [Cas73, Proposition 1] that the representation $U_i(\varpi_\pi)$ is an irreducible representation of $\mathrm{GL}_2(\mathcal{O}_F)$. From the result [Cas73, Proposition 2] we get that $c(\pi) \geq c(\varpi_\pi)$. By [Cas73, Theorem 1] we have

$$\mathrm{res}_{\mathrm{GL}_2(\mathcal{O}_F)} V_\pi = V^{(c(\pi)-1)} \oplus \bigoplus_{i \geq c(\pi)} U_i(\varpi_\pi). \tag{3.14}$$

We now describe the representation $U_i(\varpi_\pi)$ in our language. Let κ be a non-trivial character of the group $K_2(m)U_{(1,1)}(\mathcal{O}_F)$ trivial modulo

$K_2(m)U_{(1,1)}(\mathcal{O}_F) \cap B_2(m+1)$. Let $Z(\kappa)$ be a $B_2(m)$ stabilizer of κ. Any element of the group $Z(\kappa)$ is of the form (for an appropriate choice of a non-trivial character κ)

$$\begin{pmatrix} a & b \\ \varpi_F^m c & d \end{pmatrix}$$

where $a, d \in \mathcal{O}_F^\times$; $b \in \mathcal{O}_F$, $c \in \mathfrak{P}_F^m$ and $d \equiv a$ modulo $\mathfrak{P}_F$. We define a character U_η by setting

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \eta(c).$$

We then have

$$U_m(\varpi) \simeq \mathrm{ind}_{Z(\eta)}^{\mathrm{GL}_2(\mathcal{O}_F)}(U_\eta \otimes (\varpi \boxtimes \mathrm{id})).$$

Now let us resume the proof in the general case $n > 2$ the representation

$$\mathrm{ind}_{Z(\eta)}^{\mathrm{GL}_{(n+1)}(\mathcal{O}_F)}\{(ext\{\Psi^+(X_{n-1})\} \boxtimes \chi_{n+1}) \otimes U_\eta\}$$

is contained in the representation

$$\mathrm{ind}_{P_{(n-1,2)}(m)}^{\mathrm{GL}_{(n+1)}(\mathcal{O}_F)}(X_{n-1} \boxtimes U_m(\chi)) \tag{3.15}$$

where χ is given by $\prod_{i=1}^n \chi_i$ of $\mathcal{O}_F^\times$. This representation, by the theorem of Casselman (see the decomposition 3.14) is contained in the representation

$$\mathrm{ind}_{P_{(n-1,2)} \cap \mathrm{GL}_{(n+1)}(\mathcal{O}_F)}^{\mathrm{GL}_{(n+1)}(\mathcal{O}_F)}(X'_{n-1} \boxtimes \sigma)$$

where σ is a supercuspidal representation of level-zero with central character χ (see the remark below for the existence) and X'_{n-1} is the $(n-1)$ derivative of the representation

$$i_{B_n}^{\mathrm{GL}_n(F)}(\chi_{I_n}).$$

Hence irreducible sub-representations of (3.12) are atypical. This completes the proof that irreducible sub-representations of

$$\mathrm{ind}_{Z(\eta)}^{\mathrm{GL}_{n+1}(\mathcal{O}_F)}\{U_\eta \otimes \mathrm{res}_{Z(\eta) \cap M_{(n,1)}}\{\mathrm{ind}_{B_n(1)}^{\mathrm{GL}_n(\mathcal{O}_F)}(\chi_{I_n}) \boxtimes \chi_{n+1}\}\}$$

are atypical. From the decomposition

$$\begin{aligned} &\mathrm{ind}_{P(m+1)}^{\mathrm{GL}_{n+1}(\mathcal{O}_F)}\{\mathrm{ind}_{B_n(1)}^{\mathrm{GL}_n(\mathcal{O}_F)}(\chi_{I_n}) \boxtimes \chi_{n+1}\} \\ &\simeq \mathrm{ind}_{P(m)}^{\mathrm{GL}_{n+1}(\mathcal{O}_F)}\{\mathrm{ind}_{B_n(1)}^{\mathrm{GL}_n(\mathcal{O}_F)}(\chi_{I_n}) \boxtimes \chi_{n+1}\} \\ &\qquad \oplus \mathrm{ind}_{Z(\eta)}^{\mathrm{GL}_{n+1}(\mathcal{O}_F)}\{U_\eta \otimes \mathrm{res}_{Z(\eta) \cap M_{(n,1)}}\{\mathrm{ind}_{B_n(1)}^{\mathrm{GL}_n(\mathcal{O}_F)}(\chi_{I_n}) \boxtimes \chi_{n+1}\}\}. \end{aligned}$$

we get the theorem for the case where $n_r = 1$.

Remark 3.1.2. *The existence of the cuspidal representation of* $\mathrm{GL}_2(k_F)$ *with a given central character can be deduced from the explicit formula for such representations, we refer to [BH06, Theorem section 6.4]. To be precise we begin with a quadratic extension* k *of* k_F *and* θ *a character of* $k^\times$ *such that* $\theta^q \neq \theta$ *where* $q = \#k_F$*. These characters are called regular characters and for any regular character one can define a supercuspidal representation* π_θ *and conversely all supercuspidal representations are of the form* π_θ *for some regular character* θ*. The central character of* π_θ *is given by* $\mathrm{res}_{k_F^\times}(\theta)$*. Now to get a supercuspidal representation with a central character* χ *we begin with a character* χ *on* $k_F^\times$*, there are* $\#k_F + 1$ *possible extensions to* $k^\times$*. The set of characters* θ *such that* $\theta^q = \theta$ *has cardinality* $\#k_F - 1$*. Hence there exists at least one supercuspidal representation with a given central character* χ*. This shows that irreducible sub-representations of* (3.15) *are not typical and this completes the proof of the theorem in this case.*

3.1.2 The case where $n_r > 1$

By transitivity of induction we have

$$\mathrm{ind}_{P_I(m)}^{P_I(1)}(\tau_I) \simeq \mathrm{ind}_{P_I(1,m)}^{P_I(1)}\{\mathrm{ind}_{P_I(m)}^{P_I(1,m)}(\tau_I)\}.$$

We note that $P_I(1,m) \cap U_{(n-n_r+1,n_r)}$ is equal to $P_I(m) \cap U_{(n-n_r+1,n_r)}$ and $P_I(1,m) \cap \bar{U}_{(n-n_r+1,n_r)}$ is equal to $P_I(m) \cap \bar{U}_{(n-n_r+1,n_r)}$ hence lemma 2.2.6 gives the isomorphism

$$\mathrm{ind}_{P_I(1,m)}^{P_I(1)}\{\mathrm{ind}_{P_I(m)}^{P_I(1,m)}(\tau_I)\} \simeq \mathrm{ind}_{P_I(1,m)}^{P_I(1)}\{(\mathrm{ind}_{P_{I'}(m)}^{P_{I'}(1)}(\tau_{I'})) \boxtimes \tau_r)\}.$$

Splitting the representation $\mathrm{ind}_{P_{I'}(m)}^{P_{I'}(1)}(\tau_{I'})$ as $\tau_{I'} \oplus U_m^0(\tau_{I'})$ we get that

$$\mathrm{ind}_{P_I(1,m)}^{P_I(1)}\{(\mathrm{ind}_{P_{I'}(m)}^{P_{I'}(1)}(\tau_{I'})) \boxtimes \tau_r)\} \simeq \mathrm{ind}_{P_I(1,m)}^{P_I(1)}\{U_m^0(\tau_{I'}) \boxtimes \tau_r\} \oplus \mathrm{ind}_{P_I(1,m)}^{P_I(1)}(\tau_I).$$

From Frobenius reciprocity the representation τ_I occurs in $\mathrm{ind}_{P_I(1,m)}^{P_I(1)}(\tau_I)$ with multiplicity one. Let $U_{(1,m)}^0(\tau_I)$ be the complement of τ_I in $\mathrm{ind}_{P_I(1,m)}^{P_I(1)}(\tau_I)$. With this we conclude that

$$\mathrm{ind}_{P_I(m)}^{P_I(1)}(\tau_I) \simeq \mathrm{ind}_{P_I(1,m)}^{P_I(1)}\{U_m^0(\tau_{I'}) \boxtimes \tau_r\} \oplus U_{(1,m)}^0(\tau_I) \oplus \tau_I.$$

By definition $U_m(\tau_I) = \mathrm{ind}_{P_I(1)}^{\mathrm{GL}_n(\mathcal{O}_F)}(U_m^0(\tau_I))$ which shows that

$$U_m(\tau_I) \simeq \mathrm{ind}_{P_I(1,m)}^{\mathrm{GL}_{n+1}(\mathcal{O}_F)}\{U_m^0(\tau_{I'}) \boxtimes \tau_r\} \oplus \mathrm{ind}_{P_I(1)}^{\mathrm{GL}_{n+1}(\mathcal{O}_F)}(U_{(1,m)}^0(\tau_I)).$$

We observe that $P_I(1,m) \cap U_{(n-n_r+1,n_r)} = P_{(n-n_r+1,n_r)}(m) \cap U_{(n-n_r+1,n_r)}$ and $P_I(1,m) \cap \bar{U}_{(n-n_r+1,n_r)} = P_{(n-n_r+1,n_r)}(m) \cap \bar{U}_{(n-n_r+1,n_r)}$ hence lemma

2.2.6 applied to the groups $J_2 = P_I(1,m)$ and $J_1 = P_{(n-n_r+1,n_r)}(m)$ and $\lambda = U^0_m(\tau_{I'}) \boxtimes \tau_r$ gives us the isomorphism

$$\mathrm{ind}_{P_I(1,m)}^{\mathrm{GL}_{n+1}(\mathcal{O}_F)}\{U^0_m(\tau_{I'}) \boxtimes \tau_r\} \simeq \mathrm{ind}_{P_{(n-n_r+1,n_r)}(m)}^{\mathrm{GL}_{n+1}(\mathcal{O}_F)}\{U_m(\tau_{I'}) \boxtimes \tau_r\}.$$

With this we are in a place to use the induction hypothesis through the isomorphism

$$U_m(\tau_I) \simeq \mathrm{ind}_{P_{(n-n_r+1,n_r)}(m)}^{\mathrm{GL}_{n+1}(\mathcal{O}_F)}\{U_m(\tau_{I'}) \boxtimes \tau_r\} \oplus \mathrm{ind}_{P_I(1)}^{\mathrm{GL}_{n+1}(\mathcal{O}_F)}(U^0_{(1,m)}(\tau_I)). \quad (3.16)$$

By induction hypothesis $\mathrm{GL}_{n-n_r+1}(\mathcal{O}_F)$-irreducible sub-representations of $U_m(\tau_{I'})$ are atypical for the component $[M_{I'}, \sigma_{I'}]$. Now lemma 3.0.18 and the equation (3.16) reduce the proof of the theorem to showing that irreducible sub-representations of $\mathrm{ind}_{P_I(1,m)}^{\mathrm{GL}_{n+1}(\mathcal{O}_F)}(U^0_{(1,m)}(\tau_I))$ are atypical representations.

Proposition 3.1.3. *The irreducible sub-representations of*

$$\mathrm{ind}_{P_I(1)}^{\mathrm{GL}_{n+1}(\mathcal{O}_F)}(U^0_{(1,m)}(\tau_I))$$

are atypical for $m \geq 1$.

Proof. We observe that

$$\mathrm{ind}_{P_I(1,m+1)}^{P_I(1)}(\tau_I) \simeq \mathrm{ind}_{P_I(1,m)}^{P_I(1)}\{\mathrm{ind}_{P_I(1,m+1)}^{P_I(1,m)}(\tau_I)\}$$

and the decomposition (3.3) gives us the isomorphism

$$\mathrm{ind}_{P_I(1,m+1)}^{P_I(1)}(\tau_I) = \mathrm{ind}_{P_I(1,m)}^{P_I(1)}(\tau_I) \oplus \bigoplus_{\eta_k \neq \mathrm{id}} \mathrm{ind}_{P_I(1,m)}^{P_I(1)}\{(\mathrm{ind}_{Z(\eta_k)}^{P_I(1,m)}(U_{\eta_k}) \otimes \tau_I\}$$

which gives the equality

$$U^0_{(1,m+1)}(\tau_I) = U^0_{(1,m)}(\tau_I) \oplus \bigoplus_{\eta_k \neq \mathrm{id}} \mathrm{ind}_{P_I(1,m)}^{P_I(1)}\{(\mathrm{ind}_{Z(\eta_k)}^{P_I(1,m)}(U_{\eta_k})) \otimes \tau_I\}.$$

If we show that the irreducible sub-representations of

$$\mathrm{ind}_{P_I(1,m)}^{\mathrm{GL}_{n+1}(\mathcal{O}_F)}\{(\mathrm{ind}_{Z(\eta_k)}^{P_I(1,m)}(U_{\eta_k})) \otimes \tau_I\}$$

(for $\eta_k \neq \mathrm{id}$) are atypical for $[M_I, \sigma_I]$ then induction on the positive integer m completes the proof of the proposition in this case. To begin with we note that

$$\mathrm{ind}_{P_I(1,m)}^{\mathrm{GL}_{n+1}(\mathcal{O}_F)}\{\mathrm{ind}_{Z(\eta_k)}^{P_I(1,m)}(U_{\eta_k}) \otimes \tau_I\}$$
$$\mathrm{ind}_{P_I(1,m)}^{\mathrm{GL}_{n+1}(\mathcal{O}_F)}\{\mathrm{ind}_{Z(\eta_k)}^{P_I(1,m)}(U_{\eta_k} \otimes \mathrm{res}_{Z(\eta_k) \cap M_I} \tau_I)\}.$$

The representation τ_I is a level zero representation. Hence $\operatorname{res}_{Z(\eta_k)\cap M_I}\tau_I$ is isomorphic to the inflation of the representation $\operatorname{res}_{\overline{Z(\eta_k)\cap M_I}}\tau_I$ where $\overline{Z(\eta_k)\cap M_I}$ is mod-$\mathfrak{P}_F$ reduction of $Z(\eta_k)\cap M_I$. Let $A=\theta_I(\eta_k)$ where θ_I is the map defined in the paragraph just after lemma 3.0.14. The mod-$\mathfrak{P}_F$ reduction $\overline{Z(\eta_k)\cap M_I}$ is contained in $Z_{P_{I'}(k_F)\times \mathrm{GL}_{n_r}(k_F)}(A)$. If η_k is a nontrivial character then $A\neq 0$ and we can apply lemma 3.0.15. For convenience we break the proof of this proposition into subsections considering different possibilities in lemma 3.0.15.

3.1.3 Condition (1) of lemma 3.0.15

We first assume that A satisfies the condition (1) in lemma 3.0.15. There exists at least one n_j with $1\leq j\leq r$ such that the image of the projection

$$p_j : Z_{P_{I'}(k_F)\times \mathrm{GL}_{n_r}(k_F)}(A)\cap M_I \to \mathrm{GL}_{n_j}(k_F)$$

is contained in a proper parabolic subgroup of $\mathrm{GL}_{n_j}(k_F)$. Here p_j is the projection onto the j-th factor of M_I. In particular n_j is greater than 1. Let γ be an irreducible sub-representation of the restriction $\operatorname{res}_H\tau_j$ where H is the image of $M_I\cap Z(\eta_k)$ under the projection p_j. It follows from lemma 3.0.16 that there exists an irreducible non-cuspidal representation τ' of $\mathrm{GL}_{n_j}(k_F)$ such that $\tau_j\not\simeq\tau'$ and γ is contained in $\operatorname{res}_H\tau'$. Let τ' (as a representation of $\mathrm{GL}_{n_j}(\mathcal{O}_F)$ obtained by inflation) be a sub-representation of

$$\Gamma = i_{P_J(1)}^{\mathrm{GL}_{n_j}(\mathcal{O}_F)}(\kappa_J)$$

where $J=(m_1,m_2,\dots,m_t)$ is an ordered partition of the positive integer n_j and each of κ_l for $1\leq l\leq t$ is a cuspidal representation of $\mathrm{GL}_{m_l}(k_F)$. Define a representation τ_I^1 (a first modification of τ_I) of $Z(\eta_k)\cap M_I$ by setting

$$\tau_I^1 := \tau_1\boxtimes\tau_2\boxtimes\cdots\boxtimes\tau_{j-1}\boxtimes\gamma\boxtimes\tau_{j+1}\boxtimes\cdots\boxtimes\tau_r.$$

The representation

$$\operatorname{ind}_{P_I(1,m)}^{\mathrm{GL}_{n+1}(\mathcal{O}_F)}\{\operatorname{ind}_{Z(\eta_k)}^{P_I(1,m)}(U_{\eta_k}\otimes\tau_I^1)\} \tag{3.17}$$

is contained in the representation

$$\operatorname{ind}_{P_I(1,m)}^{\mathrm{GL}_{n+1}(\mathcal{O}_F)}\{\operatorname{ind}_{Z(\eta_k)}^{P_I(1,m)}(U_{\eta_k}\otimes\operatorname{res}_{Z(\eta_k)\cap M_I}\tau_I^2)\}$$

where τ_I^2 (the second modification) is the representation

$$\tau_I^1 := \tau_1\boxtimes\tau_2\boxtimes\cdots\boxtimes\tau_{j-1}\boxtimes\Gamma\boxtimes\tau_{j+1}\boxtimes\cdots\boxtimes\tau_r.$$

Observe that

$$\operatorname{ind}_{P_I(1,m)}^{\mathrm{GL}_{n+1}(\mathcal{O}_F)}\{\operatorname{ind}_{Z(\eta_k)}^{P_I(1,m)}(U_{\eta_k}\otimes\tau_I^2)\}\simeq\operatorname{ind}_{P_I(1,m)}^{\mathrm{GL}_{n+1}(\mathcal{O}_F)}\{\operatorname{ind}_{Z(\eta_k)}^{P_I(1,m)}(U_{\eta_k})\otimes\tau_I^2\}$$

The representation $\mathrm{ind}_{P_I(1,m)}^{\mathrm{GL}_n(\mathcal{O}_F)}\{\mathrm{ind}_{Z(\eta_k)}^{P_I(1,m)}(U_{\eta_k}) \otimes \tau_I^2\}$ is a sub-representation of the representation $\mathrm{ind}_{P_I(1,m+1)}^{\mathrm{GL}_n(\mathcal{O}_F)}(\tau_I^2)$ which in turn is contained in the representation $\mathrm{ind}_{P_I \cap \mathrm{GL}_n(\mathcal{O}_F)}^{\mathrm{GL}_n(\mathcal{O}_F)}(\tau_I^2)$. We denote by I_1 the refinement of the ordered partition I obtained by replacing n_j with the ordered partition $J = (m_1, m_2, \dots, m_t)$. We define κ_{I_1} a representation of $M_{I_1}(\mathcal{O}_F)$ by setting

$$\kappa_{I_1} := \tau_1 \boxtimes \cdots \boxtimes \tau_{j-1} \boxtimes \kappa_1 \boxtimes \cdots \boxtimes \kappa_t \boxtimes \tau_{j+1} \boxtimes \cdots \boxtimes \tau_r.$$

By setting these notations we now note that

$$\mathrm{ind}_{P_I \cap \mathrm{GL}_n(\mathcal{O}_F)}^{\mathrm{GL}_n(\mathcal{O}_F)}(\tau_I^2) \subset \mathrm{ind}_{P_{I_1} \cap \mathrm{GL}_n(\mathcal{O}_F)}^{\mathrm{GL}_n(\mathcal{O}_F)}(\kappa_{I_1}).$$

Since I_1 is a proper partition of I the Bushnell-Kutzko types $(P_I(1), \tau_I)$ and $(P_{I_1}(1), \kappa_{I_1})$ represent two distinct inertial classes.

3.1.4 Condition (2) of lemma 3.0.15

Let $A = \theta_I(\eta_k)$ satisfy the condition (2) in the lemma 3.0.15. In this case there exists a j with $1 \leq j < r$ such that the mod $\mathfrak{P}_F$ reduction of $Z(\eta_k) \cap M_I$ is contained in the subgroup of the form

$$\{(A_1, \dots, A_j, \dots, A_r) |\ A_i \in \mathrm{GL}_{n_i}(k_F)\ \forall\ i \in \{1, 2, \dots, r\} \text{ and } A_j = A_r\}.$$

Note that $n_j = n_r$ and we assumed that $n_r > 1$. Consider the representation $\tau_j \boxtimes \tau_r$ of $\mathrm{GL}_{n_j}(k_F) \times \mathrm{GL}_{n_r}(k_F)$ and $H = \{(A, A)\ |A \in \mathrm{GL}_{n_r}(k_F)\}$. For every irreducible sub-representation γ of $\mathrm{res}_H(\tau_j \boxtimes \tau_r)$ using lemma 3.0.17 we obtain an irreducible non-cuspidal representation $\tau_j^1 \boxtimes \tau_r^1$ such that γ is contained in $\mathrm{res}_H(\tau_j^1 \boxtimes \tau_r^1)$. Now define a representation τ_I^1 by setting

$$\tau_I^1 := \tau_1 \boxtimes \tau_2 \boxtimes \cdots \boxtimes \tau_j^1 \boxtimes \cdots \boxtimes \tau_r^1.$$

We note here that τ_I^1 may not be independent of γ in the sense that that (τ_j^1, τ_r^1) depends on the irreducible sub-representation γ of $\mathrm{res}_H(\tau_k \boxtimes \tau_r)$. Any irreducible sub-representation Γ of

$$\mathrm{ind}_{P_I(1,m)}^{\mathrm{GL}_{n+1}(\mathcal{O}_F)}\{\mathrm{ind}_{Z(\eta_k)}^{P_I(1,m)}(U_{\eta_k} \otimes \mathrm{res}_{Z(\eta_k) \cap M_I} \tau_I)\}$$

occurs as a sub-representation of some

$$\mathrm{ind}_{P_I(1,m)}^{\mathrm{GL}_{n+1}(\mathcal{O}_F)}\{\mathrm{ind}_{Z(\eta_k)}^{P_I(1,m)}(U_{\eta_k} \otimes \mathrm{res}_{Z(\eta_k) \cap M_I} \tau_I^1)\}. \tag{3.18}$$

The representation in (3.18) is contained as a sub-representation of

$$\mathrm{ind}_{P_I \cap \mathrm{GL}_{n+1}(\mathcal{O}_F)}^{\mathrm{GL}_{n+1}(\mathcal{O}_F)}(\tau_I^1). \tag{3.19}$$

Let τ_j^1 and τ_r^1 be sub-representations of $\mathrm{ind}_{P_{J_1}(1)}^{\mathrm{GL}_{n_k}(\mathcal{O}_F)}(\kappa_{J_1})$ and $\mathrm{ind}_{P_{J_2}(1)}^{\mathrm{GL}_{n_r}(\mathcal{O}_)\ F}(\kappa_{J_2})$ respectively. Let I_1 be the partition of the positive integer n obtained by replacing n_j and n_r by the partitions J_1 and J_2 in $(n_1, n_2 \ldots, n_j, \ldots, n_r)$. We denote by τ_{I_1} the representation

$$\tau_{n_1} \boxtimes \cdots \boxtimes \tau_{n_{j-1}} \boxtimes \kappa_{J_1} \boxtimes \tau_{n_{j+1}} \boxtimes \cdots \boxtimes \kappa_{J_2}$$

of $M_{I_1}(\mathcal{O}_F)$. The representation (3.19) is contained in the representation

$$\mathrm{ind}_{P_{I_1} \cap \mathrm{GL}_{n+1}(\mathcal{O}_F)}^{\mathrm{GL}_{n+1}(\mathcal{O}_F)}(\tau_{I_1}).$$

The Bushnell-Kutzko types $(P_I(1), \tau_I)$ and $(P_{I_1}(1), \tau_{I_1})$ represent two distinct inertial classes since I_1 is a proper refinement of I (see lemma 3.0.17). $\square$

This completes the proof of the proposition and also the proof of the theorem. $\square$

Chapter 4

Principal series components

We denote by B_n the Borel subgroup of $\mathrm{GL}_n(F)$ consisting of upper triangular matrices. Let T_n and U_n be the maximal torus and the unipotent radical of B_n respectively. In this chapter we will classify typical representations for the components $s = [T_n, \chi]$ where χ is a character of T_n. Let τ be a typical representation for the component s. The $\mathrm{GL}_n(\mathcal{O}_F)$-representation τ occurs in a $\mathrm{GL}_n(F)$-irreducible smooth representation π. Let B be a Borel subgroup, T the maximal torus of B and χ' be a character of T. If (T, χ') and (T_n, χ) are inertially equivalent then the representation π occurs as a sub-quotient of $i_B^{\mathrm{GL}_n(F)}(\chi'')$ where χ'' is obtained from χ' by twisting with an unramified character of T. Now to classify typical representations it is enough to say which $\mathrm{GL}_n(\mathcal{O}_F)$-irreducible sub-representations of $i_B^{\mathrm{GL}_n(F)}(\chi')$ are typical for the component $[T_n, \chi]$. Let σ be a permutation of the set $\{1, 2, \ldots, n\}$ and $\chi = \boxtimes_{i=1}^n \chi_i$ be a given characer of $T_n = \prod_{i=1}^n F^\times$. We denote by χ^σ the character $\boxtimes_{i=1}^n \chi_{\sigma(i)}$ of T_n. We observe that the pairs $(T_n, \sigma(\chi))$ and (T_n, χ) are inertially equivalent. We will use a convenient permutation σ which satisfies the condition in the following lemma. For a character χ of $F^\times$ we denote by $l(\chi)$ the level of χ, i.e. the least **positive integer m** such that $1 + \mathfrak{P}_F{}^m$ is contained in the kernel of χ.

Lemma 4.0.4. *Given any sequence of characters $x_i = \chi_i$ of $\mathcal{O}_F^\times$, there exists a permutation $\{y_i \mid 1 \le i \le n\}$ of $\{x_i \mid 1 \le i \le n\}$ such that*

$$l(y_i y_k^{-1}) \geq \max\{l(y_i y_j^{-1}), l(y_j y_k^{-1})\}.$$

for all $1 \le i \le j \le k \le n$.

Proof. For any ultrametric space (X, d) and given any n points $x_1, x_2, x_3, \ldots, x_n$ in X we may choose a permutation $y_1, y_2, \ldots, y_n$ of the sequence $\{x_i | 1 \le i \le n\}$ such that

$$d(y_i, y_k) \geq \max\{d(y_i, y_j), d(y_j, y_k)\}$$

for all $i \le j \le k$. Now apply this fact to the space X consisting of characters of $\mathcal{O}_F^\times$ and the distance function $d(\chi_1, \chi_2)$ is defined as the level $l(\chi_1 \chi_2^{-1})$ if $\chi_1 \neq \chi_2$ and 0 otherwise. We point out that this ordering is not unique in general. We refer to [How73][lemma 1] for a proof of these results. □

Remark 4.0.5. *We note that the condition $l(y_i y_k^{-1}) \geq \max\{l(y_i y_j^{-1}), l(y_j y_k^{-1})\}$ is equivalent to an equality since we always have*

$$l(y_i y_k^{-1}) \leq \max\{l(y_i y_j^{-1}), l(y_j y_k^{-1})\}.$$

Given a principal series component $[T_n, \chi]$ we choose the representative (T_n, χ^σ) where σ is a permutation such that

$$l(\chi_{\sigma(i)}\chi_{\sigma(k)}^{-1}) \geq \max\{l(\chi_{\sigma(i)}\chi_{\sigma(j)}^{-1}), l(\chi_{\sigma(j)}\chi_{\sigma(k)}^{-1})\}.$$

From now on we assume that the pair $(T_n, \boxtimes_{i=1}^n \chi_i)$ satisfies the condition

$$l(\chi_i\chi_k^{-1}) \geq \max\{l(\chi_i\chi_j^{-1}), l(\chi_j\chi_k^{-1})\} \tag{4.1}$$

for all $i \leq j \leq k$.

4.1 Construction of compact open subgroups H_m

Let $\mathcal{A} = (a_{ij})$ be a lower nilpotent matrix of size $n \times n$ such that a_{ij} is non-negative for $i > j$ and

$$a_{ki} = \max\{a_{ji}, a_{kj}\} \tag{4.2}$$

for all i, j and k such that $1 \leq i < j < k \leq n$. We denote by $J(\mathcal{A})$ the set of $n \times n$ matrices (m_{pq}) such that $m_{pq} \in \mathcal{O}_F$ for $p < q$ and $m_{pq} \in \mathfrak{P}_F^{a_{pq}}$ for $p \geq q$. As a consequence of the condition $a_{ki} = \max\{a_{ji}, a_{kj}\}$ we get two important inequalities

$$a_{i1} \geq a_{i2} \geq \cdots \geq a_{ii-1} \tag{4.3}$$

and

$$a_{j+1j} \leq a_{j+2j} \leq \cdots \leq a_{nj}. \tag{4.4}$$

The first is a consequence of $a_{ik-1} = \max\{a_{kk-1}, a_{ik}\}$ for $k < i$ and the second is a consequence of $a_{k+1j} = \max\{a_{k+1k}, a_{kj}\}$ for $j < k$.

Lemma 4.1.1. *The set $\mathcal{J}(\mathcal{A})$ is an order in $M_n(\mathcal{O}_F)$*

Proof. The set $\mathcal{J}(\mathcal{A})$ is an additive group. We now check that the set $\mathcal{J}(\mathcal{A})$ is closed under multiplication. Let (m_{ij}) and (m'_{ij}) be two matrices from $J(\mathcal{A})$. If $i > j$ then the $i \times j$ term in the product matrix $(m_{ij})(m'_{ij})$ can be split into three terms: $t_1 := m_{i1}m'_{1j} + m_{i2}m'_{2j} + \cdots + m_{ij}m'_{ji}$, $t_2 := m_{ij+1}m'_{j+1k} + \cdots + m_{ii}m'_{ij}$ and $t_3 := m_{ii+1}m'_{i+1j} + \cdots + m_{in}m'_{nj}$. The valuation of the term $m_{ik}m'_{kj}$ is greater or equal to a_{i1} for $k \leq j$. This shows that valuation of t_1 is greater or equal to $\min\{a_{i1}, a_{i2}, \ldots, a_{ij}\}$ and $\min\{a_{i1}, \ldots, a_{ij}\} \geq a_{ij}$. The valuation of the term $m_{ik}m'_{kj}$ is greater or equal to $a_{ik} + a_{kj}$ for all $j \leq k \leq i$ and $a_{ik} + a_{kj}$ is greater or equal to a_{ij}. This shows that the valuation of t_2 is greater or equal to a_{ij}. Finally the valuation of $m_{ik}m_{kj}$ is greater or equal to a_{kj} for $k > i$. The valuation of the term t_3 is greater or equal to $\min\{a_{i+1j}, \ldots, a_{nj}\}$ and $\min\{a_{i+1j}, \ldots, a_{nj}\} \geq a_{ij}$. Hence the additive group $\mathcal{J}(\mathcal{A})$ is closed under multiplication. Since $\mathcal{J}(\mathcal{A})$ is an $\mathcal{O}_F$ lattice in $M_n(F)$ we get that $\mathcal{J}(\mathcal{A})$ is an order in $M_n(\mathcal{O}_F)$. □

We denote by $J(\mathcal{A})$ the set of invertible elements of $\mathcal{J}(\mathcal{A})$. The following are a few examples of $J(\mathcal{A})$.

1. If $\mathcal{A} = 0$ then the group $J(\mathcal{A})$ is $\mathrm{GL}_n(\mathcal{O}_F)$.

2. If $\mathcal{A} = (a_{ij})$ with $a_{ij} = 1$ for $i > j$, then $J(\mathcal{A})$ is the Iwahori subgroup with respect to the standard Borel subgroup B_n.

3. Let $s = [T_n, \boxtimes_{i=1}^n \chi_i]$ be an inertial class satisfying the condition 4.1 and $\mathcal{A}_\chi$ be the lower nilpotent matrix (a_{ij}) where $a_{ij} = l(\chi_i\chi_j^{-1})$ for $i > j$.

The examples (2) and (3) satisfy Iwahori decomposition with respect to the standard Borel subgroup B_n. The next lemma concerns the Iwahori decomposition of $J(\mathcal{A})$ in general.

Let $\mathcal{A} = (a_{ij})$ be a lower nilpotent matrix such that $a_{ki} = \max\{a_{kj}, a_{ji}\}$ for $1 \leq i < j < k \leq n$. We define an ordered partition I of n by induction on the set of positive integers $m \leq n$. $I_1 := (1)$ now if we know $I_m = (n_1, n_2, \dots, n_r)$ for some $m \leq n-1$ then I_{m+1} is the partition $(n_1, n_2, \dots, n_r, 1)$ if $a_{m+1m} \neq 0$ and $(n_1, n_2, \dots, n_r + 1)$ if $a_{m+1m} = 0$. We denote by $I(\mathcal{A})$ the partition I_n.

Lemma 4.1.2. *The group $J(\mathcal{A})$ satisfies Iwahori decomposition with respect to the parabolic subgroup $P_{I(\mathcal{A})}$ and the standard Levi-subgroup $M_{I(\mathcal{A})}$. We have $J(\mathcal{A}) \cap M_{I(\mathcal{A})} = M_{I(\mathcal{A})}(\mathcal{O}_F)$, $J(\mathcal{A}) \cap U_{I(\mathcal{A})} = U_{I(\mathcal{A})}(\mathcal{O}_F)$.*

Proof. We use induction on the positive integer n. If $n = 1$ then $J(\mathcal{A})$ is $\mathcal{O}_F^\times$ and the lemma is vacuously true. We assume that the lemma is true for all positive integers less than n. Let $I(\mathcal{A})$ be the ordered partition $(n_1, n_2, \dots n_r)$. If $r = 1$ then the lemma is true by default. We suppose $r > 1$. We will show below that every element $j \in J(\mathcal{A})$ can be written as a product $u_1 j_1$ with $u_1 \in J(\mathcal{A}) \cap \bar{U}_{(n_1, n-n_1)}$ and $j_1 \in J(\mathcal{A}) \cap P_{(n_1, n-n_1)}$. Now j_1 can be written as $j_2 u_1^+$ where $u_1^+ \in U_{(n_1, n-n_1)}(\mathcal{O}_F)$ and $j_2 \in M_{(n_1, n-n_1)} \cap J(\mathcal{A})$. Now j_2 can be written as $j_3 u_2^+$ where $j_3 \in J(\mathcal{A}) \cap M_{I(\mathcal{A})}$ and $u_2^+ \in U_{I(\mathcal{A})}(\mathcal{O}_F)$. The group $J(\mathcal{A}) \cap M_{(n_1, n-n_1)}$ is equal to $\mathrm{GL}_{n_1}(\mathcal{O}_F) \times J(\mathcal{A}')$ where the nilpotent matrix $\mathcal{A}' = (a'_{ij})$ is given by $a'_{ij} = a_{i+n_1 j+n_1}$. By induction hypothesis $J(\mathcal{A}')$ satisfies Iwahori decomposition with respect to the standard parabolic subgroup $P_{I(\mathcal{A}')}$ and its Levi-subgroup $M_{I(\mathcal{A}')}$ and $I(\mathcal{A}') = (n_2, n_3, \dots, n_r)$. Let $j_3 = (j_3^0, j_3^1)$ where $j_3^0 \in \mathrm{GL}_{n_1}(\mathcal{O}_F)$ and $j_3^1 \in J(\mathcal{A}')$. Now $j_3^1 = u_3^- j_4 u_3^+$ where $u_3^- \in \bar{U}_{I(\mathcal{A}')} \cap J(\mathcal{A}')$, $u_3^+ \in U_{I(\mathcal{A}')} \cap J(\mathcal{A}')$ and $j_4 \in M_{I(\mathcal{A}')} \cap J(\mathcal{A}')$. Hence $j = u_1 u_3^- (j_3^0, j_4) u_3^+ u_2^+$ (with a slight abuse of notation the elements u_3^- and u_3^+ are considered as elements of $\bar{U}_{I(\mathcal{A})}$ and $U_{I(\mathcal{A})}$ respectively and (j_3^0, j_4) is an element of $J(\mathcal{A}) \cap M_{I(\mathcal{A})} = \mathrm{GL}_{n_1}(\mathcal{O}_F) \times (J(\mathcal{A}') \cap M_{I(\mathcal{A}')})$).

We now prove that $j \in J(\mathcal{A})$ can be written as a product $u_1 j_1$ with $u_1 \in J(\mathcal{A}) \cap \bar{U}_{(n_1, n-n_1)}$ and $j_1 \in J(\mathcal{A}) \cap P_{(n_1, n-n_1)}$. Let $j = (j_{pq})$. Let C_i^1 be the i^{th}-column of the first diagonal block (of size $n_1 \times n_1$) on the diagonal. If every entry of C_i^1 has positive valuation then, we claim that the all the entries of the i^{th} column C_i have positive valuation. Suppose the k^{th} entry j_{ki} of C_i is an unit for some $k > n_1$. This shows that a_{ki} the ki^{th}-entry of $\mathcal{A}$ is zero. Now the inequality (4.3) gives $a_{ki} \geq a_{kn_1}$ and this implies that $a_{kn_1} = 0$. Now note that $a_{kn_1} \geq a_{n_1+1n_1}$ from the inequality (4.4). This shows that $a_{n_1+1n_1}$ is zero which gives a contradiction from the definition of $I(\mathcal{A})$. We now deduce that j_{ki} is not invertible. This shows the claim. Since j is invertible we conclude that at least one entry of C_i^1 is an unit. Let $E_{ij}(c) = I_n + e_{ij}(c)$ where $e_{ij}(c)$ is the matrix with its ij entry c and all other entries 0. The left multiplication of $E_{ij}(c)$ results in the row operation $R_j + cR_i$. Since at least one entry of C_i^1 is an unit we assume that its q^{th}-entry is an unit. We can perform row operations $R_p + cR_q$ for all $p \geq n_1$ to make the p^{th}-entry trivial. We also note that the elementary matrix corresponding to this row operation also belongs to the group $J(\mathcal{A})$ (note that $q \leq n_1 \leq p$). This completes the task of making j as the product $u_1 j_1$.

The uniqueness of the Iwahori decomposition is standard. The proof is included for the completeness. If $u_1^- j_1 u_1^+ = u_2^- j_2 u_2^+$ then $(u_2^-)^{-1} u_1^- j_1 u_1^+ (u_2^+)^{-1} = j_2$. Let $u_3^- = (u_2^-)^{-1} u_1^-$ and $u_3^+ = u_1^+ (u_2^+)^{-1}$. We then have $u_3^- j_1 u_3^+ = j_2$. The equality can be rewritten as

$$u_3^- j_1 j_2^{-1} = (j_2 u_3^+ j_2^{-1})^{-1}.$$

The right hand side of the above equality is an upper block matrix with identity on diagonal blocks and the left hand side is lower block matrix. This shows that right hand side is identity matrix. Similar reasoning shows that u_3^- and u_3^+ are both identity matrices and $j_1 = j_2$. This proves the uniqueness of the Iwahori decomposition. $\square$

Let $s = [T_n, \chi]$ be an inertial equivalence class. Let m be a positive integer and $\mathcal{A}_\chi(m)$ be the lower nilpotent matrix (a_{ij}^m) where $a_{ij}^m = l(\chi_i \chi_j^{-1}) + m - 1$. As shown earlier the representative $(T_n, \chi = \boxtimes_{i=1}^n \chi_i)$ can be chosen such that

$$a_{ik} = \max\{a_{ij}, a_{jk}\}$$

for all $i < j < k$. We denote by $J_\chi(m)$ the group $J(\mathcal{A}_\chi(m))$. Note that $J_\chi(m') \subset J_\chi(m)$ for all $m' \geq m$. In our situation we have $I(\mathcal{A}_\chi(m))$ is $(1, 1, \ldots, 1)$ since none of a_{ii+1}^m are zero and hence by lemma 4.1.2 $J_\chi(m)$ satisfies Iwahori decomposition with respect to B_n.

Lemma 4.1.3. *The character $\chi = \boxtimes_{i=1}^n \chi_i$ of $T(\mathcal{O}_F)$ extends to a character of $J_\chi(1)$ such that $J_\chi(1) \cap U_n$ and $J_\chi(1) \cap \bar{U}_n$ are contained in the kernel of the extension.*

Proof. Let $m = (m_{ij})$ be an element of $J_\chi(1)$. We define $\tilde{\chi}(m) = \prod_{i=1}^n \chi_i(m_{ii})$. We verify that $\tilde{\chi}$ is a character of the group $J_\chi(1)$. This is very computational in nature. We sketch the proof here and for complete details see [Roc98, Section 3, Lemma 3.1, Lemma 3.2] or [How73, Pg 278-279]. The idea is to get an open normal subgroup U of $J_\chi(1)$ such that $J_\chi(1)/U$ is isomorphic to $T(\mathcal{O}_F)/T_\chi$ where T_χ is an open subgroup of $T(\mathcal{O}_F)$ which is contained in the kernel of χ. The subgroup U is generated by $J_\chi(1) \cap \bar{U}_n$ and $J_\chi(1) \cap U_n = U_n(\mathcal{O}_F)$. One shows that U satisfies Iwahori decomposition with respect to the Borel subgroup B_n and $U \cap T_n$ is given by $\prod_{\alpha \in \Phi} \alpha^\vee(1 + \mathfrak{P}_F^{l(\chi\alpha^\vee)})$ where Φ is the set of roots of GL_n with respect to T_n and $\alpha^\vee$ stands for the dual root. We observe that $U \cap T_n$ is contained in the kernel of χ. $\square$

We apply lemma 2.2.5 to the sequence of groups $H_m = J_\chi(m)$ for $m \geq 1$ and $\tau = \chi$ to get the isomorphism

$$\mathrm{res}_{\mathrm{GL}_n(\mathcal{O}_F)} i_{B_n}^{\mathrm{GL}_n(F)}(\chi) = \bigcup_{m \geq 1} \mathrm{ind}_{J_\chi(m)}^{\mathrm{GL}_n(\mathcal{O}_F)}(\chi).$$

We denote by $\mathcal{A}_\chi(1,m)$ the lower nilpotent matrix (a_{ij}) where $a_{ij} = l(\chi_i\chi_j^{-1})$ for $j < i < n$, $a_{nj} = l(\chi_n\chi_j^{-1}) + m - 1$. Given a lower nilpotent matrix $\mathcal{A} = (a_{ij})$ such that $a_{ki} = \max\{a_{kj}, a_{ji}\}$ we associated a compact subgroup $J(\mathcal{A})$. The condition $a_{ki} = \max\{a_{kj}, a_{ji}\}$ is a sufficient condition to define the group $J(\mathcal{A})$ in a similar recipe and this condition can be verified easily. The matrix $\mathcal{A}_\chi(1,m)$ need not satisfy this condition but we can still associate the group $J(\mathcal{A}_\chi(1,m))$ to the matrix $\mathcal{A}_\chi(1,m)$. We will prove this in the next lemma.

Lemma 4.1.4. *Let $\mathcal{J}(\mathcal{A}_\chi(1,m))$ be the set consisting of matrices (m_{ij}) such that $m_{ij} \in \mathfrak{P}_F^{a_{ij}}$ for all i,j. The set $\mathcal{J}(\mathcal{A}_\chi(1,m))$ is an order in $M_n(\mathcal{O}_F)$.*

Proof. The set $\mathcal{J}(\mathcal{A}_\chi(1,m))$ is clearly an additive group and is open. We have to verify that $\mathcal{J}(\mathcal{A}_\chi(1,m))$ is closed under multiplication. Let (m_{ij}) and (m'_{ij}) be two elements of the set $\mathcal{J}(\mathcal{A}_\chi(1,m))$. We suppose $i > j$. The ij^{th}-term of the product $(m_{ij})(m'_{ij})$ is the sum of the terms:

$$t_1 := m_{i1}m'_{1j} + m_{i2}m'_{2j} + \cdots + m_{ij}m'_{ji},$$

$$t_2 := m_{ij+1}m'_{j+1k} + \cdots + m_{ii}m'_{ij}$$

and

$$t_3 := m_{ii+1}m'_{i+1j} + \cdots + m_{in}m'_{nj}.$$

The valuation of the term t_1 is greater or equal to the minimum value among the valuation of $m_{ik}m'_{kj}$ for $1 \leq k \leq j$ and the valuation of $m_{ik}m'_{kj}$ is given by a_{ik}. If $i < n$ then $a_{ik} = l(\chi_i\chi_k^{-1})$ and $a_{ij} \leq a_{ik}$ for all $k \leq j < i$. This shows that t_1 has valuation at least a_{ij}. If $i = n$ then the values of $a_{ik} = a_{nk} = l(\chi_n\chi_k^{-1}) + m - 1 \geq l(\chi_i\chi_j^{-1}) + m - 1 = a_{ij}$ for all $k \leq j < n$. We conclude that in every possibility the valuation of t_1 is greater or equal to a_{ij}.

Consider the term t_2. The valuation of t_2 is at least the minimum among the valuation of $m_{ik}m'_{kj}$ for $j < k \leq i$. The valuation of $m_{ik}m'_{kj}$ is given by $a_{ik} + a_{kj}$ for $j < k \leq i$. If $i < n$ $a_{ik} = l(\chi_i\chi_k^{-1})$ and $a_{kj} = l(\chi_k\chi_j^{-1})$. From our assumption on the arrangement of characters χ_i for $1 \leq i \leq n$, we get that $l(\chi_i\chi_j^{-1}) = \max\{l(\chi_i\chi_k^{-1}), l(\chi_k\chi_j^{-1})\}$. At the same time $i < n$ implies $a_{ij} = l(\chi_i\chi_j^{-1})$. This shows that the valuation of the term is $m_{ik}m'_{kj}$ given by $a_{ik} + a_{kj}$ is at least a_{ij}. Consider the case $i = n$ and $a_{nk} = l(\chi_n\chi_k^{-1}) + m - 1$. Now $a_{kj} = l(\chi_k\chi_j^{-1})$ and $a_{nj} = l(\chi_n\chi_j^{-1}) + m - 1$. From the equality $l(\chi_i\chi_j^{-1}) = \max\{l(\chi_i\chi_k^{-1}), l(\chi_k\chi_j^{-1})\}$ we deduce that

$$l(\chi_i\chi_k^{-1}) + l(\chi_k\chi_j^{-1}) > l(\chi_i\chi_j^{-1})$$

and adding $m - 1$ on both sides we get $a_{ij} > a_{ik} + a_{kj}$. We conclude that the valuation of the term t_2 is at least a_{ij}.

The valuation of the term $m_{ik}m'_{kj}$ for $i < k < n$ is given by $a_{kj} = l(\chi_k\chi_j^{-1})$ which is greater or equal to a_{ij} and $a_{nj} = l(\chi_n\chi_j^{-1}) + m - 1 \geq a_{ij}$ and we conclude that the valuation of t_3 is at least a_{ij}. This shows that the valuation of $t_1 + t_2 + t_3$ is at least a_{ij} which proves our result.

□

Let $J_\chi(1, m)$ be the group of units of $\mathcal{J}(\mathcal{A}_\chi(1, m)$. We will need the structure of the representation

$$\mathrm{ind}_{J_\chi(1,m+1)}^{J_\chi(1,m)}(\mathrm{id})$$

for the proof of our main theorem. We follow the strategy already established in the previous chapter. Let $K_\chi(1, m)$ be the set of matrices (m_{ij}) such that $m_{ij} \in \mathfrak{P}_F$ for $i < j < n$ and $m_{in} \in \mathcal{O}_F$ for $i < n$, $m_{ii} \in 1 + \mathfrak{P}_F$ for $i \leq n$ and $m_{ij} \in {\mathfrak{P}_F}^{a_{ij}}$ for $i > j$ and (a_{ij}) is the matrix $\mathcal{A}_\chi(1, m)$.

Lemma 4.1.5. *The set $K_\chi(1, m)$ is a normal subgroup of $J_\chi(1, m)$.*

Proof. We first check that $K_\chi(1, m)$ is closed under matrix multiplication. Let (m_{ij}) and (m'_{ij}) be two matrices from the set $K_\chi(1, m)$. Let $i < j < n$ the ij^{th} term is the sum of

$$t_1 = m_{i1}m'_{1j} + m_{i2}m'_{2j} + \cdots + m_{ii}m'_{ij},$$

$$t_2 = m_{ii+1}m'_{i+1j} + m_{ii+2}m'_{i+2j} + \cdots + m_{ij}m'_{jj}$$

and

$$t_3 = m_{ij+1}m'_{j+1j} + \cdots + m_{in}m'_{nj}.$$

The valuation of m'_{kj} is positive for $1 < k \leq i$ hence t_1 has positive valuation. The valuation of m_{ik} is positive for $i < k \leq j$ and hence t_2 is positive. The valuation of m'_{kj} is positive for $j < k \leq n$ hence valuation of t_3 is positive. This shows that ij^{th}-term of the matrix product has positive valuation. The rest of the verifications on congruence conditions are verified in lemma 4.1.4. The existence of inverse for an element in $K_\chi(1, m)$ follows from Gaussian elimination.

Now we establish the normality of $K_\chi(1,m)$. The group $K_\chi(1,m)$ satisfies Iwahori decomposition with respect to the subgroups $P_{(n-1,1)}$ and $M_{(n-1,1)}$. We also note that $K_\chi(1,m)\cap U_{(n-1,1)}$ is equal to $J_\chi(1,m)\cap U_{(n-1,1)}$ and $K_\chi(1,m)\cap \bar{U}_{(n-1,1)}$ is equal to $J_\chi(1,m)\cap \bar{U}_{(n-1,1)}$. To check the normality of $K_\chi(1,m)$ we have to check that $J_\chi(1,m)\cap M_{(n-1,1)}$ normalizes $K_\chi(1,m)$. This is equivalent to checking that $K_\chi(1,m)\cap M_{(n-1,1)}$ is a normal subgroup of $J_\chi(1,m)\cap M_{(n-1,1)}$.

We note that $J_\chi(1,m)\cap M_{(n-1,1)} = J_{\chi'}(1)\times \mathcal{O}_F^\times$ where $\chi' = \boxtimes_{i=1}^{n-1}\chi_i$. Let p_1 be the projection of $J_\chi(1,m)\cap M_{(n-1,1)}$ onto $J_{\chi'}(1)$ and π_1 be the reduction mod $\mathfrak{P}_F$ map. Note that $K_\chi(1,m)\cap M_{(n-1,1)}$ is the kernel of $\pi_1\circ p_1$. □

From the above lemma the group $K_\chi(1,m)$ is a normal subgroup of $J_\chi(1,m)$. We also note that $J_\chi(1,m)\cap \bar{U}_{(n-1,1)}$ is contained in $K_\chi(1,m)$. From this we conclude that $J_\chi(1,m) = K_\chi(1,m)J_\chi(1,m+1)$. From the Mackey decomposition we get that

$$\mathrm{res}_{K_\chi(1,m)}\mathrm{ind}_{J_\chi(1,m+1)}^{J_\chi(1,m)}(\mathrm{id}) \simeq \mathrm{ind}_{K_\chi(1,m)\cap J_\chi(1,m+1)}^{K_\chi(1,m)}(\mathrm{id}).$$

From the definition of $K_\chi(1,m)$ we get that $K_\chi(1,m)\cap J_\chi(1,m+1) = K_\chi(1,m+1)$ and

$$\mathrm{res}_{K_\chi(1,m)}\mathrm{ind}_{J_\chi(1,m+1)}^{J_\chi(1,m)}(\mathrm{id}) \simeq \mathrm{ind}_{K_\chi(1,m+1)}^{K_\chi(1,m)}(\mathrm{id}). \tag{4.5}$$

Lemma 4.1.6. *The group $K_\chi(1,m+1)$ is a normal subgroup of $K_\chi(1,m)$.*

Proof. Since the groups $K_\chi(1,m)$ satisfy Iwahori decomposition, $K_\chi(1,m)\cap U_{(n-1,1)}$ is equal to $K_\chi(1,m+1)\cap U_{(n-1,1)}$ and $K_\chi(1,m)\cap M_{(n-1,1)}$ is equal to $K_\chi(1,m+1)\cap M_{(n-1,1)}$. We have to check that $u^-j(u^-)^{-1}$ and $u^-u^+(u^-)^{-1}$ belong to $K_\chi(1,m+1)$ for all u^-, j and u^+ in

$K_\chi(1,m)\cap \bar{U}_{(n-1,1)}$,

$K_\chi(1,m)\cap M_{(n-1,1)}$ and

$K_\chi(1,m)\cap U_{(n-1,1)}$.

respectively.

We first consider the case $u^-j(u^-)^{-1}$. We can rewrite $u^-j(u^-)^{-1}$ as $j\{j^{-1}u^-j(u^-)^{-1}\}$. Since $j\in K_\chi(1,m)\cap M_{(n-1,1)} = K_\chi(1,m+1)\cap M_{(n-1,1)}$, it is enough to show that $j^{-1}u^-j(u^-)^{-1}$ belongs to the group $K_\chi(1,m+1)$. Let j and u^- be written in their block matrix form as follows.

$$j = \begin{pmatrix} J_1 & 0\\ 0 & j_1\end{pmatrix} \quad u^- = \begin{pmatrix} 1_n & 0\\ U^- & 1\end{pmatrix}$$

The conjugation $j^{-1}u^-j(u^-)^{-1}$ in its block form is given by

$$\begin{pmatrix} 1_{n-1} & 0\\ j_1^{-1}U^-J_1 - U^- & 1\end{pmatrix}$$

Let $U^- = [u_1, u_2, \ldots, u_{n-1}]$ and $J_1 = (j_{ij})$. The k^{th} entry of the matrix $U^- J_1$ is the sum of $t_1 = u_1 j_{1k} + u_2 j_{2k} + \cdots + u_{k-1} j_{k-1k}$, $t_2 = u_k j_{kk}$ and $t_3 = u_{k+1} j_{k+1k} + \cdots + u_{n-1} j_{n-1k}$. Let $l(\chi_k \chi_n^{-1}) > 1$ then valuation of $u_t j_{tk}$ for $t < k$ is at least $l(\chi_t \chi_n^{-1}) + m - 1 + 1$ which is at least $l(\chi_k \chi_n^{-1}) + m$ this shows that valuation of the term t_1 is at least $l(\chi_k \chi_n^{-1}) + m - 1$. The valuation of $u_t a_{tk}$ for $k < t$ is at least $l(\chi_t \chi_n^{-1}) + l(\chi_t \chi_k^{-1}) + m - 1 > l(\chi_k \chi_n^{-1}) + m - 1$. This shows that $t_1 + t_2 + t_3 \equiv t_2 = u_k j_{kk} = u_2 \mod \mathfrak{P}_F{}^{l(\chi_k \chi_n^{-1}) + m}$. We note that $j_1^{-1} u_2 \equiv u_2$ hence the matrix

$$\begin{pmatrix} 1_n & 0 \\ j_1^{-1} U^- J_1 - U^- & 1 \end{pmatrix}$$

is contained in $K_\chi(1, m+1) \cap \bar{U}_{(n-1,1)}$

Let us consider the conjugation $u^- u^+ (u^-)^{-1}$. We write u^+ in the block form as

$$\begin{pmatrix} 1_n & U^+ \\ 0 & 1 \end{pmatrix}$$

The conjugated matrix $u^- u^+ (u^-)^{-1}$ is given by

$$\begin{pmatrix} 1_{n-1} - U^+ U^- & U^+ \\ -U^- U^+ U^- & U^- U^+ + 1 \end{pmatrix}.$$

Let $1_{n-1} - U^+ U^- = (u_{ij})$. The valuation of u_{ij} for $i > j$ is greater or equal to $l(\chi_n \chi_j^{-1})$ and $l(\chi_n \chi_j^{-1})$ is greater or equal to $l(\chi_i \chi_j^{-1})$. From this we conclude that $u^- u^+ (u^-)^{-1} \in K_\chi(1, m+1)$. $\square$

4.2 Preliminaries for main theorem

The inclusion map of $K_\chi(1, m) \cap \bar{U}_n$ in $K_\chi(1, m)$ induces an isomorphism of the quotient $K_\chi(1, m)/K_\chi(1, m+1)$ with the abelian quotient

$$\frac{K_\chi(1, m) \cap \bar{U}_{(n-1,1)}}{K_\chi(1, m+1) \cap \bar{U}_{(n-1,1)}}. \tag{4.6}$$

Hence the representation $\mathrm{ind}_{K_\chi(1,m+1)}^{K_\chi(1,m)}(\mathrm{id})$ splits into characters η_k with $1 \leq k \leq p$. The group $J_\chi(1, m)$ acts on these characters and let $Z(\eta_k)$ be the $J_\chi(1, m)$-stabilizer of the character η_k. From Clifford theory we get that

$$\mathrm{ind}_{J_\chi(1,m+1)}^{J_\chi(1,m)}(\mathrm{id}) \simeq \bigoplus_{\eta_{n_k}} \mathrm{ind}_{Z(\eta_{n_k})}^{J_\chi(1,m)}(U_{\eta_{n_k}}) \tag{4.7}$$

where η_{n_k} is a representative for an orbit under the action of $J_\chi(1, m)$ and $U_{\chi_{n_k}}$ is an irreducible representation of the group $Z(\eta_{n_k})$. Since

$$J_\chi(1, m) = (J_\chi(1, m) \cap M_{(n-1,1)}) K_\chi(1, m)$$

we get that $Z(\eta_k) = (Z(\eta_k) \cap M_{(n-1,1)})K_\chi(1,m)$.

The final step in our preliminaries is to understand the mod $\mathfrak{P}_F$ reduction of the group

$$Z(\eta_k) \cap M_{(n-1,1)}$$

for some non-trivial character η_k. The group $J_\chi(1,m) \cap M_{(n-1,1)}$ is equal to $J_{\chi'}(1) \times \mathcal{O}_F^\times$ acts on the quotient

$$\frac{K_\chi(1,m) \cap \bar{U}_{(n-1,1)}}{K_\chi(1,m+1) \cap \bar{U}_{(n-1,1)}} \tag{4.8}$$

by conjugation. Let j and u^- be two elements from $J_\chi(1,m) \cap M_{(n-1,1)}$ (which is $J_{\chi'}(1) \times \mathcal{O}_F^\times$ for $\chi' = \boxtimes_{i=1}^{n-1}\chi_i$) and $K_\chi(1,m) \cap \bar{U}$ respectively. We write the elements j and u^- written in their block diagonal form as

$$\begin{pmatrix} J_1 & 0 \\ 0 & j_1 \end{pmatrix} \text{ and } \begin{pmatrix} 1_2 & 0 \\ U^- & 1_2 \end{pmatrix}$$

respectively. The map $u^- \mapsto \varpi_F^{-(m-1)}U^-$ induces an isomorphism between the group (4.8) and $M_{1\times(n-1)}(k_F)$. The group $J_\chi(1) \times \mathcal{O}_F^\times$ acts through its mod $\mathfrak{P}_{PF}$ reduction-$B_{n-1}(k_F) \times k_F^\times$ on the set of matrices $M_{1\times(n-1)}(k_F)$ by setting $(b,x)(A) = xAb^{-1}$ for all $b \in B_{n-1}(k_F), x \in k_F^\times$ and $A \in M_{1\times(n-1)}(k_F)$. Hence the map $u^- \mapsto \varpi_F^{-(m-1)}U^-$ gives an $J_\chi(1) \times {\mathcal{O}_F}^\times$-equivariant map between $M_{1\times(n-1)}(k_F)$ and the group (4.8). We also have a $M_{(n-1,1)}(k_F)$-equivariant map between the group of characters of $M_{1\times(n-1)}(k_F)$ and $M_{(n-1)\times 1}(k_F)$ (see lemma 3.0.14). Hence we obtain a $J_\chi(1) \times {\mathcal{O}_F}^\times$ equivariant map between the group of characters of the quotient (4.8) and the group $M_{(n-1)\times 1}(k_F)$ where $J_\chi(1)$ acts through its mod $\mathfrak{P}_F$ reduction-$B_{(n-1)}(k_F) \times k_F^\times$ and the action is $(b,x)A = bAx^{-1}$ (see lemma 3.0.14). Hence the group $Z(\eta_k) \cap M_{(n-1,1)}$ for non-trivial η_k is equal to $Z_{B_{n-1}(k_F)\times k_F^\times}(A)$ for some non-zero matrix A in $M_{(n-1)\times 1}(k_F)$.

Let p be the projection of $B_{n-1}(k_F) \times k_F^\times$ onto the diagonal torus

$$T_{n-1}(k_F) \times k_F^\times = T_n(k_F),$$

let p_i be the i^{th} projection of $T_n(k_F)$ onto $k_F^\times$. The centralizer $Z_{B_{n-1}(k_F)\times k_F^\times}(A)$ of a non-zero matrix $A = [u_1, u_2, \ldots, u_{n-1}]^{tr}$ satisfies the following property: there exists $j < n$ such that $p_j(p(t)) = p_n(p(t))$ for all $t \in Z_{B_{n-1}(k_F)\times k_F^\times}(m)$ (see lemma 3.0.15). This shows that for any non-trivial character η_{n_k}, $Z(\eta_{n_k}) \cap T_n$ satisfies the property that

$$p_j(t) \equiv p_n(t)$$

mod $\mathfrak{P}_F$.

The character $\chi = \boxtimes_{i=1}^{n}\chi_i$ of $J_\chi(1)$ occurs with multiplicity one in the representation

$$\mathrm{ind}_{J_\chi(m)}^{J_\chi(1)}(\chi).$$

We denote by $U_m^0(\chi)$ the complement of χ in $\mathrm{ind}_{J_\chi(m)}^{J_\chi(1)}(\chi)$. We denote by $U_m(\chi)$ the representation

$$\mathrm{ind}_{J_\chi(1)}^{\mathrm{GL}_n(\mathcal{O}_F)}\{U_m^0(\chi)\}.$$

4.3 Main theorem

Theorem 4.3.1. *Let $\#k_F > 3$. The irreducible sub-representations of $U_m(\chi)$ are atypical for the component s.*

Proof. We prove the theorem by using induction on the positive integers n and m. For $n = 1$ the representation $U_m(\chi)$ is trivial and the theorem is vacuously true. Let n be a positive integer greater than one. We assume that the theorem is proved for all positive integers less than n. We will use the induction hypothesis to show the theorem for n. We note that $J_\chi(1,m)$ and $K_\chi(1,m)$ satisfy Iwahori decomposition with respect to the parabolic subgroup $P_{(n-1,1)}$ and standard Levi-subgroup $M_{(n-1,1)}$, $J_\chi(1,m)\cap U_{(n-1,1)} = K_\chi(1,m)\cap U_{(n-1,1)}$ and $J_\chi(1,m)\cap \bar{U}_{(n-1,1)} = K_\chi(1,m)\cap \bar{U}_{(n-1,1)}$. Now lemma 2.2.6 shows that $\mathrm{ind}_{J_\chi(m)\cap M_{(n-1)}}^{J_\chi(1,m)\cap M_{(n-1)}}(\chi)$ extends to a representation of $J_\chi(1,m)$ and this extension is given by

$$\mathrm{ind}_{J_\chi(m)}^{J_\chi(1,m)}(\chi).$$

If we denote by χ' the character $\boxtimes_{i=1}^{n-1}\chi_i$ of $\prod_{i=1}^{n-1} F^\times$ then we have

$$\mathrm{ind}_{J_\chi(m)\cap M_{(n-1)}}^{J_\chi(1,m)\cap M_{(n-1)}}(\chi) \simeq \mathrm{ind}_{J_{\chi'}(m)}^{J_{\chi'}(1)}(\chi')\boxtimes\chi_n.$$

We also have

$$\mathrm{ind}_{J_{\chi'}(m)}^{J_{\chi'}(1)}(\chi')\boxtimes\chi_n \simeq U_m^0(\chi')\boxtimes\chi_n\oplus\chi.$$

Combining the above isomorphisms we get that

$$\mathrm{ind}_{J_\chi(m)}^{\mathrm{GL}_n(\mathcal{O}_F)}(\chi) \simeq \mathrm{ind}_{J_\chi(1,m)}^{\mathrm{GL}_n(\mathcal{O}_F)}\{U_m^0(\chi')\boxtimes\chi_n\}\bigoplus\mathrm{ind}_{J_\chi(1,m)}^{\mathrm{GL}_n(\mathcal{O}_F)}(\chi). \tag{4.9}$$

We will use the induction hypothesis to show that $\mathrm{GL}_n(\mathcal{O}_F)$-irreducible sub-representations of

$$\mathrm{ind}_{J_\chi(1,m)}^{\mathrm{GL}_n(\mathcal{O}_F)}\{U_m^0(\chi')\boxtimes\chi_n\} \tag{4.10}$$

are atypical representations. By induction hypothesis any $\mathrm{GL}_{n-1}(\mathcal{O}_F)$-irreducible sub-representation of $U_m(\chi')$ occurs as sub-representation of some

$$i_{P_I}^{\mathrm{GL}_{n-1}(F)}(\sigma)$$

where $[T_{n-1}, \chi']$ and $[M_I, \sigma]$ are two distinct inertial classes. We now get that irreducible sub-representations of 4.10 occur as sub-representations of

$$i_{P_{I'}}^{\mathrm{GL}_n(F)}(\sigma \boxtimes \chi_n)$$

where I' is obtained from I by adding 1 at the end of the ordered partition I of $n-1$. If $I \neq (1, 1, \dots, 1)$ then the Levi sub-groups $M_{I'}$ and T_n are not conjugate and hence the inertial classes $[M_{I'}, \sigma \boxtimes \chi_n]$ and $[T_n, \chi]$ are distinct inertial classes and this proves our claim in this case. Now let $M_I = T_{n-1}$ and $\sigma = \boxtimes_{i=1}^{n-1} \sigma_i$ be the tensor factorization of T_{n-1}. Since the components $[T_{n-1}, \chi']$ and $[T_{n-1}, \sigma]$ are distinct we get a character χ_t occurring with non-zero multiplicity in the multi-set $\{\chi_1, \chi_2, \dots, \chi_{n-1}\}$ but with a different multiplicity in the multi-set $\{\sigma_1, \sigma_2, \dots, \sigma_{n-1}\}$. Adding the character χ_n to both multi-sets above keeps the multiplicities of the character χ_t distinct and this shows that $[T_n, \chi]$ and $[T_n, \sigma \boxtimes \chi_n]$ are different inertial classes.

This shows that any typical representation must occur as a sub-representation of

$$\mathrm{ind}_{J_\chi(1,m)}^{\mathrm{GL}_n(\mathcal{O}_F)}(\chi).$$

The character χ occurs with multiplicity one in the representation $\mathrm{ind}_{J_\chi(1,m)}^{J_\chi(1)}(\chi)$. We denote by $U_{1,m}^0(\chi)$ the complement of the character χ in $\mathrm{ind}_{J_\chi(1,m)}^{J_\chi(1)}(\chi)$. We denote by $U_{1,m}(\chi)$ the representation

$$\mathrm{ind}_{J_\chi(1)}^{\mathrm{GL}_n(\mathcal{O}_F)}\{U_{1,m}^0(\chi)\}.$$

We first note that

$$U_m(\chi) \simeq \mathrm{ind}_{J_\chi(1,m)}^{\mathrm{GL}_n(\mathcal{O}_F)}\{U_m^0(\chi') \boxtimes \chi_n\} \oplus U_{1,m}(\chi).$$

We already showed that the first summand on the right-hand side of the above equation has all its irreducible sub-representations atypical. We now show that irreducible sub-representations of $U_{1,m}(\chi)$ are atypical and this proves the main theorem.

We first note that

$$\mathrm{ind}_{J_\chi(1,m+1)}^{J_\chi(1)}(\chi) \simeq \mathrm{ind}_{J_\chi(1,m)}^{J_\chi(1)}\{\mathrm{ind}_{J_\chi(1,m+1)}^{J_\chi(1,m)}(\mathrm{id}) \otimes \chi\}.$$

Using the decomposition 4.7 we get that

$$\mathrm{ind}_{J_\chi(1,m+1)}^{J_\chi(1)}(\chi) \simeq \bigoplus_{\eta_{n_k}} \mathrm{ind}_{Z(\eta_{n_k})}^{J_\chi(1)}\{U_{\eta_{n_k}} \otimes \chi\}.$$

Recall that η_{n_k} is a representative for the orbit under the action of the group $J_\chi(1,m)$ on the characters η_t of $K_\chi(1,m)$ which are trivial on $K_\chi(1,m+1)$

and $Z(\eta_{n_k})$ is the $J_\chi(1,m)$-stabilizer of the character η_{n_k}. There is exactly one orbit consisting of the id character and hence

$$\mathrm{ind}_{J_\chi(1,m+1)}^{J_\chi(1)}(\chi) \simeq \mathrm{ind}_{J_\chi(1,m)}^{J_\chi(1)}(\chi) \bigoplus_{\eta_{n_k}\neq \mathrm{id}} \mathrm{ind}_{Z(\eta_{n_k})}^{J_\chi(1)}\{U_{\eta_{n_k}} \otimes \chi\}. \tag{4.11}$$

Consider the representation

$$\mathrm{ind}_{Z(\eta_{n_k})}^{J_\chi(1)}\{U_{\eta_{n_k}} \otimes \chi\}$$

for some representative $\eta_{n_k} \neq \mathrm{id}$. Now recall that $Z(\eta_{n_k}) \cap T_n$ is a subgroup of $T_n(\mathcal{O}_F) = \prod_{i=1}^n {\mathcal{O}_F}^\times$ and there exists a positive integer $j < n$ such that $p_j(t) \equiv p_n(t) \mod \mathfrak{P}_F$ for all $t \in Z(\eta_{n_k})$. Let κ be a character of $F^\times$ such that κ is ramified and $1 + \mathfrak{P}_F$ is contained in the kernel of κ. Let χ^κ be the character

$$\chi_1 \boxtimes \chi_2 \boxtimes \chi_j\kappa \boxtimes \cdots \boxtimes \chi_n\kappa^{-1}.$$

We observe that $\mathrm{res}_{Z(\eta_{n_k})}(\chi) = \mathrm{res}_{Z(\eta_{n_k})}(\chi^\kappa)$ and hence

$$\mathrm{ind}_{Z(\eta_{n_k})}^{J_\chi(1)}\{U_{\eta_{n_k}} \otimes \chi\} \simeq \mathrm{ind}_{Z(\eta_{n_k})}^{J_\chi(1)}\{U_{\eta_{n_k}} \otimes \chi^\kappa\}. \tag{4.12}$$

From the above paragraph we get that

$$U^0_{1,m+1}(\chi) \simeq U^0_{1,m}(\chi) \bigoplus_{\eta_{n_k}\neq \mathrm{id}} \mathrm{ind}_{Z(\eta_{n_k})}^{J_\chi(1)}\{U_{\eta_{n_k}} \otimes \chi\}.$$

and from the above identity we conclude that

$$U_{1,m+1}(\chi) \simeq U_{1,m}(\chi) \bigoplus_{\eta_{n_k}\neq \mathrm{id}} \mathrm{ind}_{Z(\eta_{n_k})}^{\mathrm{GL}_n(\mathcal{O}_F)}\{U_{\eta_{n_k}} \otimes \chi\}. \tag{4.13}$$

From the equation (4.12) we get that

$$\mathrm{ind}_{Z(\eta_{n_k})}^{\mathrm{GL}_n(\mathcal{O}_F)}\{U_{\eta_{n_k}} \otimes \chi\} \simeq \mathrm{ind}_{Z(\eta_{n_k})}^{\mathrm{GL}_n(\mathcal{O}_F)}\{U_{\eta_{n_k}} \otimes \chi^\kappa\}.$$

If we choose κ such that the components $[T_n, \chi]$ and $[T_n, \chi^\kappa]$ are two distinct inertial classes then we can conclude that irreducible sub-representations of

$$\mathrm{ind}_{Z(\eta_{n_k})}^{\mathrm{GL}_n(\mathcal{O}_F)}\{U_{\eta_{n_k}} \otimes \chi\}$$

are atypical and hence using the identity (4.13) recursively we get that irreducible sub-representations of $U_{1,m}(\chi)$ are atypical representations for all positive integers m.

To prove the theorem we have to justify that we can choose a character κ as in the previous paragraph. Now for any character κ non-trivial on ${\mathcal{O}_F}^\times$ (such a character exists since $\#k_F > 2$) and trivial on $1 + \mathfrak{P}_F$, the equality of the inertial classes $[T_n, \chi]$ and $[T_n, \chi^\kappa]$ implies the equality of multiplicities of χ_j in the multi-sets $\{\chi_1, \chi_2, \dots, \chi_n\}$ and $\{\chi_1, \chi_2, \dots, \chi_j\kappa, \dots, \chi_n\kappa^{-1}\}$. The equality of multiplicities implies $\chi_j\chi_n^{-1} = \kappa$. Now if $\#k_F > 3$ we have at least two non-trivial tame characters and hence we can choose κ distinct from a possibly tame character $\chi_j\chi_n^{-1}$.

$\square$

The pair $(J_\chi(1), \chi)$ is the Bushnell-Kutzko type for the component s (see [BK99, Section 8]). Hence from the above theorem we deduce that irreducible sub-representations of

$$\mathrm{ind}_{J_\chi}^{\mathrm{GL}_n(\mathcal{O}_F)}(\chi)$$

are precisely the typical representations for the component $s = [T_n, \chi]$ and $\#k_F > 3$. Moreover we have

Corollary 4.3.2. *Let $\#k_F > 3$. Let τ be a typical representation for the component $s = [T, \chi]$ then*

$$\dim_{\mathbb{C}} \mathrm{Hom}_{\mathrm{GL}_n(\mathcal{O}_F)}(\tau, i_{B_n}^{\mathrm{GL}_n(F)}(\chi)) = \dim_{\mathbb{C}} \mathrm{Hom}_{\mathrm{GL}_n(\mathcal{O}_F)}(\tau, \mathrm{ind}_{J_\chi}^{\mathrm{GL}_n(\mathcal{O}_F)}(\chi))$$

Remark 4.3.3. *When $\#k_F = 2$ and $n = 2$ Henniart showed in [BM02][A.2.6, A.2.7] that the Bushnell-Kutzko type for the component $s = [T_2, \chi_1 \boxtimes \chi_2]$, $\chi_1\chi_2^{-1} \neq \mathrm{id}$ has two typical representations one given by*

$$\mathrm{ind}_{J_\chi(1)}^{\mathrm{GL}_2(\mathcal{O}_F)}(\chi)$$

and the other representation turns out to be the (it can be shown a priori that there is a unique complement (see [Cas73][Proposition 1(b)])) complement of $\mathrm{ind}_{J_\chi(1)}^{\mathrm{GL}_2(\mathcal{O}_F)}(\chi)$ in $\mathrm{ind}_{J_\chi(2)}^{\mathrm{GL}_2(\mathcal{O}_F)}(\chi)$. But for $\#k_F > 2$ and $n > 3$ we expect that typical representations are precisely the irreducible sub-representations of

$$\mathrm{ind}_{J_\chi(1)}^{\mathrm{GL}_n(\mathcal{O}_F)}(\chi).$$

For $\#k_F = 2$ and $n > 2$ a typical representation may not be contained in the above representation as shown by Henniart for the case of $\mathrm{GL}_2(F)$.

Chapter 5

The inertial class with Levi-subgroup of type (n,1)

Let V and V_1 be two F-vector spaces of dimensions $n > 1$ and 1 respectively. Let P be the parabolic subgroup of $GL(V \oplus V_1)$ fixing the flag $V \subset V \oplus V_1$. We denote by M its Levi-subgroup fixing the decomposition $V \oplus V_1$ and hence we have $M = GL(V) \times GL(V_1)$. In this chapter we are interested in the classification of typical representations for the component $[M, \sigma \boxtimes \chi]$ where σ is a cuspidal representation of $GL(V)$ and χ is a character of $GL(V_1)$. Let (J^0, λ) be a maximal simple (Bushnell-Kutzko's) type contained in the representation σ. We recall certain important features of this type for our purpose.

5.1 Bushnell-Kutzko semi-simple type

We denote by A the algebra $\mathrm{End}_F(V)$. Let $[\mathfrak{A}, l, 0, \beta]$ be a simple strata in A defining the maximal simple type (J^0, λ). We denote by B the algebra of endomorphisms commuting with $E = F[\beta]$. Let $\mathfrak{B} = \mathfrak{A} \cap B$. We denote by $\mathfrak{P}$ and $\mathfrak{Q}$ the radicals of $\mathfrak{A}$ and $\mathfrak{B}$ respectively. Given any hereditary order $\mathfrak{A}$, we define the filtration $U^i(\mathfrak{A})$ by setting

$$U^i(\mathfrak{A}) = 1_n + \mathfrak{P}^i$$

for all $i \geq 1$ and $U^0(\mathfrak{A})$ is the set of units of $\mathfrak{A}$. The type (J^0, λ) is called maximal if $\mathfrak{B}$ is a maximal hereditary order in B. The group J^0 contains $U^0(\mathfrak{B})$. There is a normal subgroup J^1 such that $J^1 \cap U^0(\mathfrak{B}) = U^1(\mathfrak{B})$ and

$$\frac{U^0(\mathfrak{B})}{U^1(\mathfrak{B})} \simeq \frac{J^0}{J^1}.$$

The group $U^0(\mathfrak{B})/U^1(\mathfrak{B})$ is a general linear group of a vector space over a finite field. The representation λ is an irreducible representation which is is given by a tensor product $\kappa \otimes \rho$ where κ is a representation of J^0, called β-extension (see [BK93, Chapter 5, Definition 5.2.1]) and ρ is a cuspidal representation of $U^0(\mathfrak{B})/U^1(\mathfrak{B})$ (considered as a representation of J^0 through the quotient J^0/J^1). We refer to [BK93, Chapter 5] for complete details of these constructions. For the precise definition and description see [BK93, chapter 5, Definition 5.5.10].

We fix the following conventions. Let e and f be the ramification index and inertial index of E. We fix a lattice chain $\mathcal{L}$ defining the order $\mathfrak{B}$. Let

$\mathfrak{A}$ be the hereditary order defined by the lattice chain $\mathcal{L}$. We fix a $\mathcal{O}_E$-basis $(w_1, w_2, \dots w_{n/ef})$ for the lattice chain $\mathcal{L}$ and then a $\mathcal{O}_F$-basis for $\mathcal{O}_E w_i$ for $1 \leq i \leq ef$. (see [BK93, Chapter 1, 1.1.7]). We can now extract a F-basis $(v_1, v_2, \dots v_{n+1})$ for the vector space $V \oplus V_1$ where V_1 is a one dimensional vector space over F. In this basis we write all our endomorphisms as matrices of $M_{n+1}(F)$. This also provides $J^0 \subset \mathrm{GL}_n(\mathcal{O}_F)$.

Let I be the ordered partition $(n, 1)$ of $n+1$. We are interested in the classification of typical representations for the component $[M_I, \sigma \boxtimes \chi]$. From the lemma 2.2.7 it is enough to classify the typical representations for the component $s = [M_I, \sigma \boxtimes \mathrm{id}]$.

To classify typical representations for the component $s = [M_I, \sigma \boxtimes \mathrm{id}]$ it is enough to examine which $\mathrm{GL}_{n+1}(\mathcal{O}_F)$-irreducible sub-representations of

$$\mathrm{res}_{\mathrm{GL}_{n+1}(\mathcal{O}_F)}\, i_{P_I}^{\mathrm{GL}_{n+1}(F)}(\sigma \boxtimes \mathrm{id})$$

are typical for the component $[M, \sigma \boxtimes \mathrm{id}]$. Let τ be the unique typical representation contained in the representation σ. It follows from lemma 2.2.4 that the representation

$$\mathrm{ind}_{P_I \cap \mathrm{GL}_{n+1}(\mathcal{O}_F)}^{\mathrm{GL}_{n+1}(\mathcal{O}_F)}(\tau \boxtimes \mathrm{id})$$

has a complement in

$$\mathrm{res}_{\mathrm{GL}_{n+1}(\mathcal{O}_F)}\, i_{P_I}^{\mathrm{GL}_{n+1}(F)}(\sigma \boxtimes \chi)$$

whose irreducible sub-representations are atypical.

Now we have to look for typical representations occurring in the representation

$$\mathrm{ind}_{P_I \cap \mathrm{GL}_{n+1}(\mathcal{O}_F)}^{\mathrm{GL}_{n+1}(\mathcal{O}_F)}(\tau \boxtimes \mathrm{id}).$$

For this purpose we will define the groups H_m for $m \geq N_0$ for some positive integer N_0, $\cap_{m \geq N_0} H_m = P_I(\mathcal{O}_F)$, H_m has Iwahori decomposition with respect to P_I and its Levi-subgroup M_I and $\tau \boxtimes \mathrm{id}$ admits an extension to H_{N_0} with $H_{N_0} \cap \bar{U}_I$ and $H_{N_0} \cap U_I$ in the kernel of this extension. The construction of H_m would give us

$$\mathrm{ind}_{P_I \cap \mathrm{GL}_{n+1}(\mathcal{O}_F)}^{\mathrm{GL}_{n+1}(\mathcal{O}_F)}(\tau \boxtimes \mathrm{id}) \simeq \bigcup_{m \geq N_0} \mathrm{ind}_{H_m}^{\mathrm{GL}_{n+1}(\mathcal{O}_F)}(\tau \boxtimes \mathrm{id}).$$

Before we start this construction it is instructive to first examine the Bushnell-Kutzko semi-simple type for the component $[M_I, \sigma \boxtimes \mathrm{id}]$.

Let us recall some standard material required from [BK99]. First let us begin with lattice sequences. A lattice sequence is a function Λ from $\mathbb{Z}$ to the set of $\mathcal{O}_F$-lattices in a F-vector space V such that $\Lambda(n+1) \subseteq \Lambda(n)$ for $n \in \mathbb{Z}$

and there exists an $e(\Lambda) \in \mathbb{Z}$ such that $\Lambda(n + e(\Lambda)) = \mathfrak{P}_F \Lambda(n)$ for all $n \in \mathbb{Z}$. A lattice chain is a lattice sequence with the strict inclusion between $\Lambda(n+1)$ and $\Lambda(n)$ for all $n \in \mathbb{Z}$. One extends the function Λ to the set of real numbers by setting

$$\Lambda(r) = \Lambda(-[-r])$$

for all $r \in \mathbb{R}$ and $[x]$ is the greatest integer less than or equal to x. Given two lattice sequences Λ_1 and Λ_2 in the vector spaces V_1 and V_2 over F, Bushnell and Kutzko defined the direct sum say Λ of Λ_1 and Λ_2, a lattice sequence, in the vector space $V_1 \oplus V_2$. Let $e = \operatorname{lcm}(e(\Lambda_1), e(\Lambda_2))$ Then

$$\Lambda(er) = \Lambda_1(e_1 r) \oplus \Lambda(e_2 r).$$

Given a lattice sequence Λ in a vector space V one can define a filtration $\{a_r(\Lambda) \mid r \in \mathbb{R}\}$ on the algebra $\operatorname{End}_F(V)$ given by the equation

$$a_r(\Lambda) = \{x \in \operatorname{End}_F(V) |\ x\Lambda(i) \subseteq \Lambda(i + r)\ \forall\ i \in \mathbb{Z}\}.$$

We also define $u_r(\Lambda)$ for $r > 0$ and $r \in \mathbb{Z}$ to be $1 + a_r(\Lambda)$ and $u_0(\Lambda)$ is the group of units in the order $a_0(\Lambda)$.

Let (J_s, λ_s) be the Bushnell-Kutzko type for the component

$$[\mathrm{GL}_n(F) \times \mathrm{GL}_1(F), \sigma \boxtimes \mathrm{id}].$$

The group J_s satisfies Iwahori decomposition with respect to the parabolic subgroup P_I and the Levi-subgroup M_I. Let us recall that we have the stratum $[\mathfrak{A}, l, 0, \beta]$ defining the simple type (J^0, λ) for the inertial class $[\mathrm{GL}_n(F), \sigma]$. The order $\mathfrak{A}$ is defined by a lattice chain Λ_1 with values in sub-lattices of $\mathcal{O}_F^n$. We denote by Λ_2 the lattice chain defined by $\Lambda_2(i) = \mathfrak{P}_F^i$ for all $i \in \mathbb{Z}$. Then we have

1. $J_s \cap U_{(n,1)} = u_0(\Lambda_1 \oplus \Lambda_2) \cap U_{(n,1)}$.
2. $J_s \cap M$ is $J^0 \times \mathcal{O}_F^\times$.
3. $J_s \cap \bar{U}_{(n,1)} = u_{l+1}(\Lambda_1 \oplus \Lambda_2) \cap \bar{U}_{(n,1)}$.
4. The restriction of λ_s to $J_s \cap M$ is isomorphic to $\lambda \boxtimes \mathrm{id}$, the groups $J_s \cap \bar{U}_{(n,1)}$ and $J_s \cap U_{(n,1)}$ are contained in the kernel of λ_s.

We refer to [BK99][Section 8, paragraph 8.3.1] for the construction of the above Bushnell-Kutzko's type.

Now we make an explicit calculation of the terms $u_{l+1}(\Lambda_1 \oplus \Lambda_2) \cap \bar{U}_{(n,1)}$ and $u_0(\Lambda_1 \oplus \Lambda_2) \cap U_{(n,1)}$. Note that the periodicity of the direct sum $\Lambda_1 \oplus \Lambda_2$ is the least common multiple of the periodicity of the two lattice sequences Λ_1 and Λ_2. We hence deduce that the periodicity of the lattice sequence Λ is e

where e is the period of the lattice chain Λ_1. Let t be an integer such that $0 \leq t \leq e-1$ and L_0 be the free $\mathcal{O}_F$ module $\mathcal{O}_F^{n/e}$. The lattice chain Λ_1 is given by :

$$\Lambda_1(t) = (L_0 \oplus L_0 \oplus \cdots \oplus L_0) \oplus (\varpi_F L_0 \oplus \varpi_F L_0 \oplus \cdots \oplus \varpi_F L_0)$$

where the L_0 is repeated $e-t$ times and $\varpi_F L_0$ is repeated t times in the first and second direct summand respectively. Hence the lattice chain Λ is given by

$$\Lambda(0) = \Lambda_1(0) \oplus \Lambda_2(0) = (L_0 \oplus L_0 \oplus \cdots \oplus L_0) \oplus \mathcal{O}_F$$

and

$$\Lambda(t) = \Lambda_1(t) \oplus \Lambda_2(t/e) = (L_0 \oplus L_0 \oplus \cdots \oplus L_0) \oplus (\varpi_F L_0 \oplus \varpi_F L_0 \oplus \cdots \oplus \varpi_F L_0) \oplus \mathfrak{P}_F\,.$$

for $0 \leq t \leq e-1$. We observe that the lattice sequence Λ is a lattice chain and the units of the hereditary order $a_0(\Lambda)$ corresponding to Λ, in the notation of chapter 3, are given by $P_J(1)$ where $J = (n/e, n/e, \ldots (e-1 \text{ times}) \ldots, n/e, n/e+1)$.

We note that $u_0(\Lambda) \cap U_{(n,1)} = U_{(n,1)}(\mathcal{O}_F)$. Let $\bar{\mathfrak{n}}_{(n,1)}$ be the lower nilpotent matrices of the type $(n,1)$ i.e the Lie-algebra of $\bar{U}_{(n,1)}$. We have the identity

$$u_{l+1}(\Lambda) \cap \bar{U}_{(n,1)} = 1 + (a_{l+1}(\Lambda) \cap \bar{\mathfrak{n}}_{(n,1)}).$$

Let $l+1 = el' + r$ where $0 \leq r < e$. Then from the observation that Λ is a lattice chain of periodicity e we deduce that

$$a_{l+1}(\Lambda) \cap \bar{\mathfrak{n}}_{(n,1)} = \varpi_F^{l'}(a_r(\Lambda) \cap \bar{\mathfrak{n}}_{(n,1)}).$$

Finally it remains to calculate the group $a_r(\Lambda) \cap \bar{\mathfrak{n}}_{(n,1)}$. We note that $a_r(\Lambda) \cap \bar{\mathfrak{n}}_{(n,1)}$ is the following set

$$\{x \in M_{n+1}(F) \cap \bar{\mathfrak{n}}_{(n,1)} \mid x\Lambda(i) \subseteq \Lambda(i+r) \ \forall\ i \in \mathbb{Z}\}.$$

For $r \geq 1$ the matrix $A = [M_1, M_2, \ldots, M_e, 0]$ in $\bar{\mathfrak{n}}_{(n,1)}$ (M_i is a matrix of type $1 \times n/e$ for $1 \leq i \leq e$) belongs to the set $a_r(\Lambda) \cap \bar{\mathfrak{n}}_{(n,1)}$ if and only if the following conditions are satisfied.

1. $M_i \in \varpi_F^2 M_{1\times e}(\mathcal{O}_F)$ for $i \leq r-1$ and
2. $M_i \in \varpi_F M_{1\times e}(\mathcal{O}_F)$ for $i > r-1$.

If $r = 0$ and $e > 1$ then we know that $M_i \in \varpi_F M_{1\times n/e}(\mathcal{O}_F)$ for $1 \leq i \leq e-1$ and $M_e \in M_{1\times n/e}(\mathcal{O}_F)$. If $r = 0$ and $e = 1$ then we have $A \in M_{1\times n}(\mathcal{O}_F)$. This completes the description of the Bushnell-Kutzko semi-simple type.

5.2 Preliminaries.

Let m be a positive integer and $P_I(m)$ be the inverse image of the group $P_I(\mathcal{O}_F\,/\,\mathfrak{P}_F{}^m)$ under the mod-$\mathfrak{P}_F{}^m$ reduction of $\mathrm{GL}_{n+1}(\mathcal{O}_F)$. There exists a positive integer N_1 such that the principal congruence sub-group of level N_1 is contained in the kernel of the representation τ. The representation $\tau \boxtimes \mathrm{id}$ of $M_I(\mathcal{O}_F\,/\,\mathfrak{P}_F{}^m)$ now extends to a representation of $P_I(m)$ by inflation for all $m > N_1$. We note that $P_I(m) \cap \bar{U}_I$ and $P_I(m) \cap U_I$ are both contained in the kernel of this extension. Now applying lemma 2.2.5 to the sequence of groups $P_I(m)$ and $\tau \boxtimes \mathrm{id}$ for $m \geq N_1$ we get that

$$\mathrm{ind}_{P_I \cap \mathrm{GL}_{n+1}(\mathcal{O}_F)}^{\mathrm{GL}_{n+1}(\mathcal{O}_F)}(\tau \boxtimes \mathrm{id}) \simeq \bigcup_{m \geq N_1+1} \mathrm{ind}_{P_I(m)}^{\mathrm{GL}_{n+1}(\mathcal{O}_F)}(\tau \boxtimes \mathrm{id}).$$

This concludes that the typical representations occur as sub-representations of

$$\mathrm{ind}_{P_I(m)}^{\mathrm{GL}_{n+1}(\mathcal{O}_F)}(\tau \boxtimes \mathrm{id})$$

for some positive integer $m \geq N_1 + 1$.

For making Mackey decompositions easier and other reasons, it is convenient to work with a smaller subgroup $P_I^0(m)$ of $P_I(m)$. We now modify the representation

$$\mathrm{ind}_{P_I(m)}^{\mathrm{GL}_{n+1}(\mathcal{O}_F)}(\tau \boxtimes \mathrm{id}).$$

We recall that $K_n(p)$ is the principal congruence subgroup of level p of $\mathrm{GL}_n(\mathcal{O}_F)$. The group J^0 contains the group $U^{[l/2]+1}(\mathfrak{A})$ ($U^{[l/2]+1}(\mathfrak{A}) \subset H^0 \subset J^0$). The representation λ restricted to the group $U^{[l/2]+1}(\mathfrak{A})$ is a direct sum of the same character ψ_β which is trivial on the group $U^{l+1}(\mathfrak{A})$. We also recall the notation that $l+1 = el' + r$ where $0 \leq r \leq e-1$. We note that $U^{l+1}(\mathfrak{A}) = 1_n + \varpi_F^{l'}\mathfrak{P}_{\mathfrak{A}}^r$. If $r = 0$ then $K_n(1) \subset \mathfrak{P}_{\mathfrak{A}}^r$. If $r > 1$ then from the formulas [BK93][2.5.2] we get that $K_n(2) \subset \mathfrak{P}_{\mathfrak{A}}^r$ for $0 \leq r < e$. This shows that the representation λ is trivial on $K_n(N_s)$ where N_s is given by:

Notation 5.1. *From now we fix $N_s = [(l+1)/e] + 1$ if $r = 0$ and $e > 1$, if $r = 0$ and $e = 1$ then $N_s = l + 1$ and $N_s = [(l+1)/e] + 2$ if $r \geq 1$.*

Let π be the projection map

$$P_I(\mathcal{O}_F) \to M_I(\mathcal{O}_F).$$

For $m \geq N_s$ we denote by $P_I^0(m)$ the group $K_{n+1}(m)\pi^{-1}(J^0 \times \mathcal{O}_F^\times)$. Since $K_{n+1}(m) \cap P_I \subset \pi^{-1}(J^0 \times \mathcal{O}_F^\times)$ the group $P_I^0(m)$ satisfies Iwahori decomposition with respect to the subgroup P_I and its Levi-subgroup M_I i.e we have

$$P_I^0(m) = (P_I^0(m) \cap U_I)(P_I^0(m) \cap M_I)(P_I^0(m) \cap \bar{U}_I)$$

where $P_I^0(m) \cap U_I$ is $U_I(\mathcal{O}_F)$, $P_I^0(m) \cap M_I$ is $J^0 \times \mathcal{O}_F^\times$ and $(P_I^0(m) \cap \bar{U}_I)$ is $K_{n+1}(m) \cap \bar{U}_I$. We observe that $\lambda \boxtimes \mathrm{id}$ extends to a representation of $P^0(m)$ for all $m \geq N_s$. Now the representation $\tau \boxtimes \mathrm{id}$ of $\mathrm{GL}_n(\mathcal{O}_F) \times \mathcal{O}_F^\times$ is isomorphic to

$$\{\mathrm{ind}_{J^0}^{\mathrm{GL}_n(\mathcal{O}_F)}(\lambda)\} \boxtimes \mathrm{id}.$$

Applying lemma 2.2.6 to the groups $J_1 = P_I(m)$ and $J_2 = P_I^0(m)$ and $\lambda = \lambda \boxtimes \mathrm{id}$ we get that

$$\mathrm{ind}_{P_I(m)}^{\mathrm{GL}_n(\mathcal{O}_F)}(\tau \boxtimes \mathrm{id}) \simeq \mathrm{ind}_{P_I^0(m)}^{\mathrm{GL}_n(\mathcal{O}_F)}(\lambda \boxtimes \mathrm{id})$$

for all $m \geq N_s$. Hence we have

$$\mathrm{ind}_{P_I \cap \mathrm{GL}_{n+1}(\mathcal{O}_F)}^{\mathrm{GL}_{n+1}(\mathcal{O}_F)}(\tau \boxtimes \mathrm{id}) \simeq \bigcup_{m \geq N_s} \mathrm{ind}_{P_I^0(m)}^{\mathrm{GL}_{n+1}(\mathcal{O}_F)}(\lambda \boxtimes \mathrm{id}).$$

Now a typical representation occurs as a sub-representation of

$$\mathrm{ind}_{P_I^0(m)}^{\mathrm{GL}_{n+1}(\mathcal{O}_F)}(\lambda \boxtimes \mathrm{id})$$

for some $m \geq N_s$.

As we did in the previous chapters we first have to understand the representation

$$\mathrm{ind}_{P_I^0(m+1)}^{P_I^0(m)}(\mathrm{id})$$

for $m \geq N_s$. The strategy is fairly standard by now. We will first consider a convenient normal subgroup $K_I(m)$ of $P_I(m)$ such that $P_I^0(m)$ is equal to $K_I(m)P_I^0(m+1)$ and $K_I(m) \cap P_I^0(m+1) = K_I(m+1)$ for $m \geq N_s$. For $m \geq N_s$ we define $K_I(m)$ to be the group $K_{n+1}(m)\pi^{-1}(K_n(N_s) \times (1 + \mathfrak{P}_F^{N_s}))$. This group does satisfy the above two properties.

Lemma 5.2.1. *The group $K_I(m)$ is a normal subgroup of $P_I^0(m)$ and $K_I(m+1)$ is a normal subgroup of $K_I(m)$ for all $m \geq N_s$.*

Proof. By definition of the groups $K_I(m)$ we have $K_I(m) \cap U_I = P_I^0(m) \cap U_I$ and $K_I(m) \cap \bar{U}_I = P_I^0(m) \cap \bar{U}_I$. To show the normality of $K_I(m)$ in $P_I(m)$ we have to verify that $P_I^0(m) \cap M_I$ normalize the group $K_I(m)$. But $P_I^0(m) \cap M_I$ normalizes the group $K_I(m) \cap U_I = U_I(\mathcal{O}_F)$ and $K_I(m) \cap \bar{U}_I = \bar{U}_I(\varpi_F^m \mathcal{O}_F)$. The group $K_I(m) \cap M_I$ is a normal subgroup of $M_I(\mathcal{O}_F)$ and hence $P_I^0(m) \cap M_I$ normalizes $K_I(m) \cap M_I$. This shows the first part of the lemma.

Since $K_I(m) \cap P_I = K_I(m+1) \cap P_I$, we have to check that $K_I(m) \cap \bar{U}_I$ normalizes the group $K_I(m+1)$. We note that $\bar{U}_I$ is abelian hence we have to check that the conjugations $u^- j (u^-)^{-1}$ and $u^- u^+ (u^-)^{-1}$ belong to the group $K_I(m+1)$ for all $u^- \in K_I(m) \cap \bar{U}_I$, $j \in K_I(m+1) \cap M_I = K_I(m) \cap M_I$ and

$u^+ \in K_I(m+1) \cap U_I = U_I(\mathcal{O}_F)$. Let us begin with the element $u^- j (u^-)^{-1}$. We have $u^- j (u^-)^- = j\{j^{-1} u^- j (u^-)^{-1}\}$. Let

$$j = \begin{pmatrix} J_1 & 0 \\ 0 & j_1 \end{pmatrix} \quad u^- = \begin{pmatrix} 1_n & 0 \\ U^- & 1 \end{pmatrix}$$

be the block diagonal form of j and u^-; $J_1 \in K_n(N_s), j_1 \in 1 + \mathfrak{P}_F{}^{N_s}$ and $U^- \in \varpi_F^m M_{1\times n}(\mathcal{O}_F)$. The element $j^{-1} u^- j (u^-)^{-1}$ is of the form

$$\begin{pmatrix} 1_n & 0 \\ j_1^{-1} U^- J_1 - U^- & 1 \end{pmatrix}$$

We note that the matrix $j_1^{-1} U^- J_1 - U^-$ belongs to $\varpi_F^{m+1} M_{1\times n}(\mathcal{O}_F)$. This shows that $j^{-1} u^- j (u^-)^{-1} \in K_I(m+1) \cap \bar{U}_I$. Hence we get that

$$u^- j (u^-)^- j\{j^{-1} u^- j (u^-)^{-1}\} \in K_I(m+1).$$

We now consider the conjugation $u^- u^+ (u^-)^{-1}$. We write u^+ in its block matrix form as

$$\begin{pmatrix} 1_n & U^+ \\ 0 & 1 \end{pmatrix}$$

where $U^+ \in M_{n\times 1}(\mathcal{O}_F)$. Now the conjugation $u^- u^+ (u^-)^{-1}$ in the block matrix from is as follows

$$\begin{pmatrix} 1_n - U^+ U^- & U^+ \\ -U^- U^+ U^- & U^- U^+ + 1 \end{pmatrix}.$$

Since $U^- U^+ U^- \in \varpi^{m+1} M_{1\times n}(\mathcal{O}_F)$, we conclude that $u^- u^+ (u^-)^{-1} \in K_I(m+1)$. This ends the proof of this lemma. $\square$

We use Mackey decomposition to the double coset decomposition $P_I^0(m) = K_I(m) P_I^0(m+1)$ to get that

$$\mathrm{res}_{K_I(m)} \mathrm{ind}_{P_I^0(m+1)}^{P_I^0(m)}(\mathrm{id}) \simeq \mathrm{ind}_{K_I(m+1)}^{K_I(m)}(\mathrm{id}).$$

It follows from Iwahori decomposition that the inclusion of $K_I(m) \cap \bar{U}_I$ in $K_I(m)$ induces an isomorphism between $K_I(m)/K_I(m+1)$ and the abelian group

$$\frac{K_I(m) \cap \bar{U}_I}{K_I(m+1) \cap \bar{U}_I}.$$

Hence the representation $\mathrm{ind}_{K_I(m+1)}^{K_I(m)}(\mathrm{id})$ decomposes as a direct sum of characters η_k for $1 \le k \le p$ where η_k is trivial on $K_I(m+1)$. The group $P_I^0(m)$

acts on these characters and let η_{n_k} be the set of representatives for the orbits under this action. We also denote by $Z(\eta)$ the $P_I^0(m)$-stabilizer of the character η. Now Clifford theory gives us the isomorphism

$$\mathrm{ind}_{P_I^0(m+1)}^{P_I^0(m)}(\mathrm{id}) = \bigoplus_{\eta_{n_k}} \mathrm{ind}_{Z(\eta_{n_k})}^{P_I^0(m)}(U_{\eta_{n_k}}) \tag{5.1}$$

where η_{n_k} is a representative for the action of $P_I^0(m)$ on the set of characters η_k and $U_{\eta_{n_k}}$ is an irreducible representation of $Z(\eta_{n_k})$.

Now we have to bound the group $Z(\eta)$. We note that $P_I^0(m)$ is equal to $(P_I^0(m) \cap M_I)K_I(m)$ and hence $Z(\eta) = (Z(\eta) \cap M_I)K_I(m)$. To bound the group $Z(\eta)$ we can only need to control $Z(\eta) \cap M_I$. Let $u^- \in K_I(m) \cap \bar{I}_I$ and

$$\begin{pmatrix} 1_n & 0 \\ U^- & 1 \end{pmatrix}$$

be the block form of u^- where U^- is a matrix in $\varpi_F^m M_{1\times n}(\mathcal{O}_F)$. The map $u^- \mapsto \varpi_F^{-m} U^-$ induces an $M_I(\mathcal{O}_F)$-equivariant isomorphism between $M_{1\times n}(k_F)$ and the quotient

$$\frac{K_I(m) \cap \bar{U}_I}{K_I(m+1) \cap \bar{U}_I}.$$

We also have an $M_I(\mathcal{O}_F)$-equivariant isomorphism between $M_{n\times 1}(k_F)$ and $\widehat{M_{1\times n}(k_F)}$ (see lemma 3.0.14). We note that $P_I^0(m) \cap M_I = J^0 \times \mathcal{O}_F^\times$.

Let η be a non-trivial character of $K_I(m)$ which is trivial on $K_I(m+1)$. We will now bound the subgroup $Z(\eta) \cap (U^0(\mathfrak{B}) \times \mathcal{O}_F^\times)$ for $\eta \neq \mathrm{id}$ and this will be enough for our purpose. Since we have a $M_I(\mathcal{O}_F)$-equivariant isomorphism between the group of characters on the quotient $K_I(m)/K_I(m+1)$ with $M_{n\times 1}(k_F)$, we can as well study the group $Z(A) \cap (U^0(\mathfrak{B}) \times \mathcal{O}_F^\times)$ where $Z(A)$ is the $M_I(\mathcal{O}_F)$-stabilizer of a non-zero matrix $A \in M_{n\times 1}(k_F)$. The action of the group $M_I(\mathcal{O}_F)$ factorizes through $K_n(1) \times (1 + \mathfrak{P}_F)$ from which we conclude that $(1_n + \mathfrak{D}^e) \times (1 + \mathfrak{P}_F)$ is contained in the kernel of the action of $M_I(\mathcal{O}_F)$. This reduces our situation to the following setting. The group $\mathrm{GL}_n(k_F) \times k_F^\times$ acts on $M_{n\times 1}$ by setting

$$(g_1, g_2)A = g_1 A g_2^{-1}$$

where $g_1 \in \mathrm{GL}_n(k_F)$, $g_2 \in k_F^\times$ and $A \in M_{n\times 1}(k_F)$. If we fix a $\mathcal{O}_E$-basis as in the previous paragraph then a k_F-basis for the vector space

$$(\mathcal{O}_E/\varpi_F\mathcal{O}_E)^{n/ef} = (\mathcal{O}_E/\mathfrak{P}_E^e)^{n/ef}$$

we get the inclusion

$$\mathrm{GL}_{n/ef}(\mathcal{O}_E \,/\, \mathfrak{P}_E{}^e) = U^0(\mathfrak{B})/U^e(\mathfrak{B}) \hookrightarrow \mathrm{GL}_n(k_F).$$

We are interested in the mod $\mathfrak{P}_E$ reduction of the first projection of

$$Z_{\mathrm{GL}_{n/ef}(\mathcal{O}_E\,/\,\mathfrak{P}_E{}^e)\times k_F^\times}(A)$$

for some non-zero matrix A in $M_{n\times 1}(k_F)$ (recall that $\lambda = \kappa\otimes\rho$ where ρ is a cuspidal representation of $\mathrm{GL}_{n/ef}(k_E) = U^0(\mathfrak{B})/U^1(\mathfrak{B})$). **We put** $n_0 = n/ef$.

Let ϖ_E be a uniformizer of $\mathcal{O}_E$. Let N be the operator on the k_E-vector space $V := (\mathcal{O}_E/\mathfrak{P}_E^e)^{n_0}$ given by

$$N(v) = \varpi_E.v.$$

Since $\mathcal{O}_E/\mathfrak{P}_E^e = k_E \oplus k_E\overline{\varpi_E} \oplus k_E\overline{\varpi_E}^2 \oplus \cdots \oplus k_E\overline{\varpi_E}^{e-1}$, we obtain a decomposition of $V = V_1 \oplus V_2 \oplus \cdots \oplus V_e$ such that N restricted to V_i is an isomorphism onto V_{i+1} for $i < e$ and N acts trivially on V_e. The mod $\mathfrak{P}_E$-reduction of V is the projection onto the first factor V_1. Any $k_E[N]$-linear map T is determined by its restriction to the space V_1. Given a map $T \in \mathrm{Hom}_{k_E}(V_1, V)$ we obtain an extension $\tilde{T} \in \mathrm{End}_{k_E[N]}(V)$ by setting

$$\tilde{T}(v) = N^{(i-1)}T(N^{-(i-1)}v)$$

for all $v \in V_i$ and $1 \leq i \leq e$. The map $T \mapsto \tilde{T}$ gives us an isomorphism of vector spaces

$$\mathrm{Hom}_{k_E}(V_1, V) \simeq \mathrm{End}_{k_E[N]}(V, V). \tag{5.2}$$

We may write $V = V_1 \oplus NV$. This shows that the mod $\mathfrak{P}_E$ reduction map say π_E is given by sending $\tilde{T}$ to $p_1 \circ \tilde{T}_{|V_1}$ where p_1 is the projection onto the first factor of the direct sum $V_1 \oplus V_2 \oplus \cdots \oplus V_e$. Now $\mathrm{End}_{k_E}(V_1)$ is a subspace of $\mathrm{Hom}_{k_E}(V_1, V)$ and mod $\mathfrak{P}_E$ reduction of $\widetilde{\mathrm{End}_{k_E}(V_1)}$ (the image of $\mathrm{End}_{k_E}(V_1)$ under the map $T \mapsto \tilde{T}$) is identity on $\mathrm{End}_{k_E}(V_1)$. Hence $\mathrm{Aut}_{k_E[N]}(V)$ is the semi-direct product $\widetilde{\mathrm{Aut}_{k_E}(V_1)}\ker(\pi_E)$.

Now we have the embedding of $\mathrm{GL}_{n_0}(\mathcal{O}_E/\mathfrak{P}_E^e)$ in $\mathrm{GL}_n(k_F)$ by considering V as a k_F-vector space. Let P be a parabolic subgroup fixing the flag $\mathcal{F}^i = \oplus_{j=1}^{i} V_i$ and M be its Levi-subgroup fixing the decomposition $V_1 \oplus V_2 \oplus \cdots \oplus V_e$. Now $\widetilde{\mathrm{Aut}_{k_E}(V_1)}$ diagonally embeds in M and $\ker(\pi_E)$ is a subgroup of the radical of P. The group $\mathrm{GL}_n(k_F) \times k_F^\times$ acts on $M_{n\times 1}(k_F)$ by the map $(g_1, g_2)A \mapsto g_1 A g_2^{-1}$ where $g_1 \in \mathrm{GL}_n(k_F)$, $g_2 \in k_F^\times$ and $A \in M_{n\times 1}(k_F)$. We now have the action of $\mathrm{GL}_{n_0}(\mathcal{O}_E/\mathfrak{P}_E^e) \times k_F^\times$ on $M_{n\times 1}(k_F)$ by restriction from $\mathrm{GL}_n(k_F) \times k_F^\times$. We are interested in

$$(\pi_E \times \mathrm{id})\{Z_{\mathrm{GL}_{n_0}(\mathcal{O}_E/\mathfrak{P}_E^e)\times k_F^\times}(A)\}$$

for some $A \in M_{n\times 1}(k_F)\backslash\{0\}$. We first look at $Z_{P\times k_F^\times}(A)$. Let (A_{ij}) be an element of P in its block form. Let $(A_1, A_2, \ldots, A_e)^{tr}$ be the block form of

A where A_j is a block of size $1 \times n_0$. If k is the largest positive integer such that $A_k \neq 0$ and $A_k = 0$ then we get that $A_{kk}A_k a^{-1} = A_k$ for all $((A_{ij}), a) \in Z_{P \times k_F^\times}(A)$. Hence $\{A_{kk} \mid ((A_{ij}), a) \in Z_{P \times k_F^\times}(A)\}$ is contained in a proper parabolic subgroup of $\mathrm{Aut}_{k_F}(V_k)$. We now conclude that

$$(\pi_E \times \mathrm{id})\{Z_{\mathrm{GL}_{n_0}(\mathcal{O}_E/\mathfrak{P}_E^e) \times k_F^\times}(A)\}$$

is a subgroup of $H \times k_F^\times$ where H is a subgroup of $\mathrm{Aut}_{k_E}(V_1)$ whose image under the inclusion map $\mathrm{Aut}_{k_E}(V_1) \hookrightarrow \mathrm{Aut}_{k_F}(V_1)$ is contained in a proper k_F-parabolic subgroup of $\mathrm{Aut}_{k_F}(V_1)$.

We recall the following proposition due to Paskunas (see [Pas05, Definition 6.2, lemma 6.5, Proposition 6.8]).

Proposition 5.2.2. *Let V be a k_E-vector space with (finite) dimension greater than one. Let ρ be a cuspidal representation of $\mathrm{Aut}_{k_E}(V)$. Let H be a subgroup of $\mathrm{Aut}_{k_E}(V)$ such that the image of H under the inclusion map $\mathrm{Aut}_{k_E}(V) \hookrightarrow \mathrm{Aut}_{k_F}(V)$ is contained in a proper parabolic subgroup of $\mathrm{Aut}_{k_F}(V)$. For every H-irreducible sub-representation ξ of $\mathrm{res}_H(\rho)$ there exists an irreducible representation ρ' of $\mathrm{Aut}_{k_E}(V)$ such that $\rho' \not\simeq \rho$ and $\mathrm{Hom}_H(\xi, \rho') \neq 0$.*

Going back to $Z(\eta) \cap (U^0(\mathfrak{B}) \times \mathcal{O}_F^\times)$, for $n_0 > 1$, we get that for every irreducible sub-representation ξ of

$$\mathrm{res}_{Z(\eta)\cap(U^0(\mathfrak{B})\times\mathcal{O}_F^\times)}((\kappa \otimes \rho) \boxtimes \mathrm{id})$$

there exists an irreducible representation ρ' of $U^0(\mathfrak{B})/U^1(\mathfrak{B})$ such that

$$\mathrm{Hom}_{Z(\eta)\cap(U^0(\mathfrak{B})\times\mathcal{O}_F^\times)}(\xi, (\kappa \otimes \rho') \boxtimes \mathrm{id}) \neq 0.$$

For the case $n_0 = 1$ and $\#k_F > 2$, we have to look at

$$(\pi_E \times \mathrm{id})\{Z_{\mathcal{O}_E/\mathfrak{P}_E^e{}^\times \times k_F^\times}(A)\} \tag{5.3}$$

for some nonzero matrix $A \in M_{n \times 1}(k_F)$. We notice that the group (5.3) is of the form $\{(a, a) | a \in k_F^\times\}$ if $k_E = k_F$. Let k_E be a proper extension of k_F. If (a, b) be an element of the centralizer (5.3) then $aA_k b^{-1} = A_k$ (A_k is defined in the previous paragraph). This shows that a lies in a proper parabolic subgroup of $\mathrm{GL}_f(k_F)$. This shows that the group (5.3) is of the form $\{(a, b) \; |a \in \mathbb{F}^\times, b \in k_F^\times\}$ where $\mathbb{F}$ is a proper sub-field of k_E. In the first case we consider a non-trivial character η of $U^0(\mathfrak{B})/U^1(\mathfrak{B}) = k_F^\times$. We observe that

$$\mathrm{res}_{Z_{J^0 \times \mathcal{O}_F^\times}(A)}(\lambda\eta \boxtimes \eta^{-1}) \simeq \mathrm{res}_{Z_{J^0 \times \mathcal{O}_F^\times}(A)}(\lambda \boxtimes \mathrm{id})$$

and moreover $[M, \sigma \boxtimes \mathrm{id}]$ and $[M, \sigma' \boxtimes \eta^{-1}]$ are two distinct inertial classes for any cuspidal representation σ' containing $(J^0, \lambda \otimes \eta)$.

In the second case where k_E is a proper extension of k_F, we consider a non-trivial character η of $k_E^\times$ which is trivial on $\mathbb{F}^\times$. We note that

$$\mathrm{res}_{Z_{J^0 \times \mathcal{O}_F^\times}(A)}(\lambda\eta \boxtimes \mathrm{id}) \simeq \mathrm{res}_{Z_{J^0 \times \mathcal{O}_F^\times}(A)}(\lambda \boxtimes \mathrm{id})$$

and moreover $[M, \sigma \boxtimes \mathrm{id}]$ and $[M, \sigma' \boxtimes \mathrm{id}]$ are two distinct inertial classes for any cuspidal representation σ' containing $(J^0, \lambda \otimes \eta)$.
With this we finish our preliminaries.

5.3 Main result

In this section we will prove the main result of this chapter. By Frobenius reciprocity we get that $\lambda \boxtimes \mathrm{id}$ occurs with multiplicity one in $\mathrm{ind}_{P_I^0(m)}^{P_I^0(N_s)}(\lambda \boxtimes \mathrm{id})$ for all $m \geq N_s$. We denote by $U_m^0(\lambda \boxtimes \mathrm{id})$ the complement of $\lambda \boxtimes \mathrm{id}$ in $\mathrm{ind}_{P_I^0(m)}^{P_I^0(N_s)}(\lambda \boxtimes \mathrm{id})$. We use the notation $U_m(\lambda \boxtimes \mathrm{id})$ for the representation

$$\mathrm{ind}_{P_I^0(N_s)}^{\mathrm{GL}_{n+1}(\mathcal{O}_F)}\{U_m^0(\lambda \boxtimes \mathrm{id})\}.$$

Theorem 5.3.1. *Let $\#k_F > 2$. The $\mathrm{GL}_{n+1}(\mathcal{O}_F)$-irreducible sub-representations of $U_m(\lambda \boxtimes \mathrm{id})$ are atypical for the component $[\mathrm{GL}_n(F) \times F^\times, \sigma \boxtimes \mathrm{id}]$ for all $m \geq N_s$.*

Proof. We prove the theorem by induction on the positive integer $m \geq N_s$. For $m = N_s$ the representation $U_m(\lambda \boxtimes \mathrm{id})$ is trivial hence the theorem is vacuously true. We suppose the theorem is true for some positive integer $m > N_s$ we will show the same for $m+1$. We first note that

$$\mathrm{ind}_{P_I^0(m+1)}^{\mathrm{GL}_{n+1}(\mathcal{O}_F)}(\lambda \boxtimes \mathrm{id}) \simeq \mathrm{ind}_{P_I^0(m)}^{\mathrm{GL}_{n+1}(\mathcal{O}_F)}\{\mathrm{ind}_{P_I^0(m+1)}^{P_I^0(m)}(\mathrm{id}) \otimes (\lambda \boxtimes \mathrm{id})\}.$$

From the decomposition (5.1) we get that

$$\mathrm{ind}_{P_I^0(m+1)}^{\mathrm{GL}_{n+1}(\mathcal{O}_F)}(\lambda \boxtimes \mathrm{id}) \simeq \bigoplus_{\eta_{n_k}} \mathrm{ind}_{Z(\eta_{n_k})}^{\mathrm{GL}_{n+1}(\mathcal{O}_F)}\{(\lambda \boxtimes \mathrm{id}) \otimes U_{\eta_{n_k}}\}.$$

Since there is a unique orbit of characters η_k consisting of the identity we get that

$$\mathrm{ind}_{P_I^0(m+1)}^{\mathrm{GL}_{n+1}(\mathcal{O}_F)}(\lambda \boxtimes \mathrm{id}) \simeq \mathrm{ind}_{P_I^0(m)}^{\mathrm{GL}_{n+1}(\mathcal{O}_F)}(\lambda \boxtimes \mathrm{id}) \oplus \bigoplus_{\eta_{n_k} \neq \mathrm{id}} \mathrm{ind}_{Z(\eta_{n_k})}^{\mathrm{GL}_{n+1}(\mathcal{O}_F)}\{(\lambda \boxtimes \mathrm{id}) \otimes U_{\eta_{n_k}}\}. \tag{5.4}$$

Let Γ be an irreducible sub-representation of

$$\mathrm{ind}_{Z(\eta_{n_k})}^{\mathrm{GL}_{n+1}(\mathcal{O}_F)}\{(\lambda \boxtimes \mathrm{id}) \otimes U_{\eta_{n_k}}\}. \tag{5.5}$$

We have two cases $n_0 = 1$ and $n_0 > 1$. If $n_0 = 1$ we have seen that we can find a non-trivial character η of $k_E^\times = U^0(\mathfrak{B})/U^1(\mathfrak{B})$ such that

$$\mathrm{ind}_{Z(\eta_{n_k})}^{\mathrm{GL}_{n+1}(\mathcal{O}_F)}\{(\lambda \boxtimes \mathrm{id}) \otimes U_{\eta_{n_k}}\} \simeq \mathrm{ind}_{Z(\eta_{n_k})}^{\mathrm{GL}_{n+1}(\mathcal{O}_F)}\{(\lambda\eta \boxtimes \eta^{-1}) \otimes U_{\eta_{n_k}}\}$$

or

$$\mathrm{ind}_{Z(\eta_{n_k})}^{\mathrm{GL}_{n+1}(\mathcal{O}_F)}\{(\lambda \boxtimes \mathrm{id}) \otimes U_{\eta_{n_k}}\} \simeq \mathrm{ind}_{Z(\eta_{n_k})}^{\mathrm{GL}_{n+1}(\mathcal{O}_F)}\{(\lambda\eta \boxtimes \mathrm{id}) \otimes U_{\eta_{n_k}}\}.$$

Hence in this case the irreducible sub-representations of

$$\mathrm{ind}_{Z(\eta_{n_k})}^{\mathrm{GL}_{n+1}(\mathcal{O}_F)}\{(\lambda \boxtimes \mathrm{id}) \otimes U_{\eta_{n_k}}\}$$

are atypical representations.

Now consider the case $n_0 > 1$. In this case there exists an irreducible representation ξ of $p_1(Z(\eta) \cap (U^0(\mathfrak{B}) \times \mathcal{O}_F^\times))$ such that Γ is a sub-representation of

$$\mathrm{ind}_{Z(\eta_{n_k})}^{\mathrm{GL}_{n+1}(\mathcal{O}_F)}\{((\xi \otimes \kappa) \boxtimes \mathrm{id}) \otimes U_{\eta_{n_k}}\}. \tag{5.6}$$

Now proposition 5.2.2 gives us an irreducible representation $\rho' \not\simeq \rho$ of $U^0(\mathfrak{B})$ obtained by inflation of an irreducible representation of $U^0(\mathfrak{B})/U^1(\mathfrak{B})$ such that ξ is contained in ρ'. Now the representation 5.6 is a sub-representation of

$$\mathrm{ind}_{Z(\eta_{n_k})}^{\mathrm{GL}_{n+1}(\mathcal{O}_F)}\{((\rho' \otimes \kappa) \boxtimes \mathrm{id}) \otimes U_{\eta_{n_k}}\}.$$

The above representation is contained in

$$\mathrm{ind}_{P_I^0(m+1)}^{\mathrm{GL}_{n+1}(\mathcal{O}_F)}(\rho' \boxtimes \kappa) \boxtimes \mathrm{id}).$$

The above representation by lemma 2.2.6 is isomorphic to the representation

$$\mathrm{ind}_{P_I(m+1)}^{\mathrm{GL}_{n+1}(\mathcal{O}_F)}(\tau' \boxtimes \mathrm{id}) \tag{5.7}$$

where τ' is

$$\mathrm{ind}_{J^0}^{\mathrm{GL}_n(\mathcal{O}_F)}(\rho' \boxtimes \kappa).$$

We will show that irreducible sub-representations of (5.7) are atypical for the component

$$[\mathrm{GL}_{n+1}(F), \sigma \boxtimes \mathrm{id}].$$

Any irreducible sub-representation of (5.7) occurs as a sub-representation of

$$\mathrm{ind}_{P_I(m)}^{\mathrm{GL}_{n+1}(\mathcal{O}_F)}(\gamma \boxtimes \mathrm{id})$$

where γ is an irreducible sub-representation of τ'. Now γ is contained in an irreducible smooth representation say π of $\mathrm{GL}_n(F)$. By Frobenius reciprocity this is possible only if the representation $\rho' \otimes \kappa$ of J^0 is contained in π. We have two possible situations either ρ' is cuspidal or otherwise. If ρ' is cuspidal then we can say that π is a supercuspidal representation such that $\pi \not\simeq \sigma$ hence the representation

$$\mathrm{ind}_{P_I(m+1)}^{\mathrm{GL}_{n+1}(\mathcal{O}_F)}(\gamma \boxtimes \mathrm{id})$$

occurs in

$$\mathrm{res}_{\mathrm{GL}_{n+1}(\mathcal{O}_F)}\, i_{P_I}^{\mathrm{GL}_{n+1}(F)}(\pi \boxtimes \mathrm{id})$$

at the same time $[\mathrm{GL}_n(F) \times F^\times, \pi \boxtimes \mathrm{id}] \neq [\mathrm{GL}_n(F) \times F^\times, \sigma \boxtimes \mathrm{id}]$. This shows that irreducible sub-representations of 5.7 are atypical representations.

If ρ' is not cuspidal the representation $\rho' \boxtimes \kappa$ is still irreducible [BK93, Chapter 5, Proposition 5.3.2(3)]. If $(J^0, \rho' \boxtimes \kappa)$ is contained in a smooth irreducible representation π then π also contains a simple-type $(J_1^0, \rho_1 \boxtimes \kappa_1)$ which is not maximal [BK93, Chapter 8, 8.3.5] (we also refer to the article [BH13, Lemma 2, Proposition 1] for quick reference). From this we conclude that π is not a cuspidal representation hence (5.7) is contained in a

$$\mathrm{res}_{\mathrm{GL}_{n+1}(\mathcal{O}_F)}\, i_{P_J}^{\mathrm{GL}_{n+1}(F)}(\sigma_J)$$

where J is a strict refinement of the partition I. Hence we get that the irreducible sub-representations of (5.7) are atypical representations. $\square$

The above theorem shows that typical representations occur as sub-representations of

$$\mathrm{ind}_{P_I^0(N_s)}^{\mathrm{GL}_{n+1}(\mathcal{O}_F)}(\lambda \boxtimes \mathrm{id}).$$

The above representation may still contain atypical representations. We will indeed show that this is the case and complete the classification.

The first observation is that the (the semi-simple type after fixing J^0) Bushnell-Kutzko type J_s for $s = [M_I, \sigma \boxtimes \mathrm{id}]$ is contains the group $P^0(N_s)$. Hence we will try to decompose the representation

$$\mathrm{ind}_{P_I^0(N_s)}^{J_s}(\mathrm{id}). \tag{5.8}$$

We also note that $P^0(N_s) \cap P_I = J_s \cap P_I$. Let $l + 1 = el' + r$ where $0 \leq r < e$. If $e = 1$ then $J_s = P_I^0(N_s)$ hence we have nothing to analyse further the theorem 5.3.1 completes the classification of typical representations. From now we assume that $e > 1$. We will first verify that the group $U_I(\mathcal{O}_F)$ acts trivially on the representation (5.8).

Let u^+ and u^- be two matrices from $J_s \cap U_I = U_I(\mathcal{O}_F)$ and $J_s \cap \bar{U}_I$ respectively. Let u^+ and u^- in block form be written as

$$\begin{pmatrix} 1_n & U^+ \\ 0 & 1 \end{pmatrix} \text{ and } \begin{pmatrix} 1_n & 0 \\ U^- & 1 \end{pmatrix}.$$

The block form of the conjugation $u^-u^+(u^-)^{-1}$ is given by

$$\begin{pmatrix} 1_n - U^+U^- & U^+ \\ -U^-U^+U^- & U^-U^+ + 1 \end{pmatrix}.$$

Now $U^- \in a_{l+1}(\Lambda) \cap \bar{\mathfrak{n}}_I = \varpi_F^{l'}(a_r(\Lambda) \cap \bar{\mathfrak{n}}_I)$. If $r \geq 1$ the valuation of each entry of a matrix in $a_r(\Lambda) \cap \bar{\mathfrak{n}}_I$ is at least one. This shows that the valuation of each entry in $U^-U^+U^-$ is at least $l' + 2$ this shows that the conjugation $u^-u^+(u^-)^{-1}$ lies in the group $P^0(N_s)$. If $r = 0$ and $l' = 0$ we are in the case where σ is a level-zero cuspidal representation and in this case $J_s = P_I^0(N_s)$. If $r = 0$ and $l' > 0$ then valuation of each entry in $U^-U^+U^-$ has valuation $2l' > l' + 1$ and hence $u^-u^+(u^-)^{-1} \in P_I^0(N_s)$. Hence the group $U_I(\mathcal{O}_F)$ acts trivially on the representation (5.8).

From the Iwahori decomposition of the group J_s we get that J_s is equal to $(J_s \cap \bar{P}_I)P_I^0(N_s)$. Hence we get that

$$\mathrm{res}_{J_s \cap \bar{P}_I} \mathrm{ind}_{P_I^0(N_s)}^{J_s}(\mathrm{id}) \simeq \mathrm{ind}_{P_I^0(N_s) \cap \bar{P}_I}^{J_s \cap \bar{P}_I}(\mathrm{id}).$$

Note that $J_s \cap \bar{P}_I$ is a semi-direct product of the groups $(J_s \cap M_I)$ and $(J_s \cap \bar{U}_I)$. Let η_k for $1 \leq k \leq t$ (we mean counting them with their multiplicity, but in our case the multiplicity is one) be all the characters of the group $J_s \cap \bar{U}_I$ which are trivial on the group $P_I^0(N_s) \cap \bar{U}_I$. The group $J_s \cap \bar{P}_I$ acts on these characters and let $\{\eta_{k_p}\}$ be a set of representatives for the orbits under this action. We denote by $Z(\eta_{k_p})$ the $J_s \cap \bar{P}_I$ stabiliser of the character η_{k_P}. Now Clifford theory gives the decomposition

$$\mathrm{ind}_{P_I^0(N_s) \cap \bar{P}_I}^{J_s \cap \bar{P}_I}(\mathrm{id}) \simeq \bigoplus_{\eta_{k_p}} \mathrm{ind}_{Z(\eta_{k_p})}^{J_s \cap \bar{P}_I}(U_{\eta_{k_p}})$$

where $U_{\eta_{k_p}}$ is an irreducible representation of $Z(\eta_{k_p})$. We note that the character id occurs with a multiplicity one in the list of characters η_k.

The representation $U_{\eta_{k_p}}$ is the isotypic component of the character η_{k_p} in the representation

$$\mathrm{ind}_{P_I^0(N_s) \cap \bar{P}_I}^{J_s \cap \bar{P}_I}(\mathrm{id}).$$

which naturally has the action of $Z(\eta_{k_p})$. Now if K_s is the kernel of the representation (5.8) then $K_s \cap Z(\eta_{k_p})$ acts trivially on $U_{\eta_{k_p}}$. Hence we can extend the representation $U_{\eta_{k_p}}$ to the group $Z(\eta_{k_P})K_s$ such that K_s acts trivially on the entered representation. Now consider the representation

$$\pi = \mathrm{ind}_{Z(\eta_{k_P})K_s}^{J_s} U_{\eta_{k_p}}.$$

Note that $K_s \cap \bar{P}_I$ is contained in the group $Z(\eta_{k_p}) \cap \bar{P}_I$ and moreover $U_I(\mathcal{O}_F)$ is contained in K_s hence $J_s = (J_s \cap \bar{P}_I)Z(\eta_{k_p})K_s$ hence from Mackey decomposition we have

$$\mathrm{res}_{J_s \cap \bar{P}_I} \mathrm{ind}_{Z(\eta_{k_P})K_s}^{J_s} U_{\eta_{k_p}} \simeq \mathrm{ind}_{Z(\eta_{k_P})K_s \cap (J_s \cap \bar{P}_I)}^{J_s \cap \bar{P}_I}(U_{\eta_{k_p}}) \simeq \mathrm{ind}_{Z(\eta_{k_p})}^{J_s \cap \bar{P}_I}(U_{\eta_{k_p}}).$$

We hence have

$$\mathrm{ind}_{P_I^0(N_s)}^{J_s}(\mathrm{id}) \simeq \bigoplus_{\eta_{k_p}} \mathrm{ind}_{Z(\eta_{k_p})K_s}^{J_s} U_{\eta_{k_p}}. \tag{5.9}$$

Now using the decomposition (5.9) we get the decomposition

$$\mathrm{ind}_{P_I^0(N_s)}^{\mathrm{GL}_{n+1}(\mathcal{O}_F)}(\lambda \boxtimes \mathrm{id}) \simeq \bigoplus_{\eta_{k_p}} \mathrm{ind}_{Z(\eta_{k_p})K_s}^{\mathrm{GL}_{n+1}(\mathcal{O}_F)}\{U_{\eta_{k_p}} \otimes (\lambda \boxtimes \mathrm{id})\}.$$

Note that the character id occurs with multiplicity one among the characters η_k and the fact that $Z(\mathrm{id})K_s = (J_s \cap \bar{P}_I)K_s = J_s$ implies the following isomorphism

$$\mathrm{ind}_{P_I^0(N_s)}^{\mathrm{GL}_{n+1}(\mathcal{O}_F)}(\lambda\boxtimes\mathrm{id}) \simeq \mathrm{ind}_{J_s}^{\mathrm{GL}_{n+1}(\mathcal{O}_F)}(\lambda\boxtimes\mathrm{id}) \oplus \bigoplus_{\eta_{k_p}\neq\mathrm{id}} \mathrm{ind}_{Z(\eta_{k_p})K_s}^{\mathrm{GL}_{n+1}(\mathcal{O}_F)}\{U_{\eta_{k_p}}\otimes(\lambda\boxtimes\mathrm{id})\}. \tag{5.10}$$

Lemma 5.3.2. *Let $\#k_F > 2$ and η_{k_p} be a non-trivial character. The irreducible sub-representations of*

$$\mathrm{ind}_{Z(\eta_{k_p})K_s}^{\mathrm{GL}_{n+1}(\mathcal{O}_F)}\{U_{\eta_{k_p}} \otimes (\lambda \boxtimes \mathrm{id})\}$$

are atypical.

Proof. We observe that $Z(\eta_{k_p}) = (Z(\eta_{k_P}) \cap M_I)(J_s \cap \bar{U}_I)$. This shows that we have to bound the group $Z(\eta_{k_P}) \cap M_I$ for $\eta_{k_p} \neq \mathrm{id}$. Recall that η_k for $1 \leq k \leq t$ are the characters on the quotient group

$$\frac{(J_s \cap \bar{U}_I)}{(P_I^0(N_s) \cap \bar{U}_I)} \tag{5.11}$$

Now let u^- be a matrix from the group $J_s \cap \bar{U}_I$. In the block form the matrix u^- is of the form

$$\begin{pmatrix} 1_n & 0 \\ U^- & 1 \end{pmatrix}$$

where $U^- = [M_1, M_2, \ldots, M_e]$, M_i is a matrix of size $(1 \times n/e)$. Let $\delta = N_s - 1$ then the map Φ

$$[M_1, M_2, \ldots, M_e] \mapsto [\varpi_F^\delta M_1, \varpi_F^\delta M_2, \ldots, \varpi_F^\delta M_e]$$

identifies the quotient 5.11 with the subspace of $M_{1\times n}(k_F)$. We have a $M_I(\mathcal{O}_F)$ equivariant map from group of characters of $M_{1\times n}(k_F)$ and $M_{n\times 1}(k_F)$ moreover $M_I(\mathcal{O}_F)$ acts through the quotient $M_I(k_F)$ (see lemma 3.0.14). The map Φ commutes with the action of $M_I \cap J_s$ since Φ is none other than conjugation by an element from the $Z(M_I)$ (The centre of M_I). Now the group $(U^0(\mathfrak{B}) \times \mathcal{O}_F^\times) \subset J_s \cap M_I$ acts on a non-zero matrix A in the space $M_{n\times 1}(k_F)$ through the quotient by $(1_n + \mathfrak{Q}^e) \times (1 + \mathfrak{P}_F)$. Now recall that we denote by

π_E by mod $\mathfrak{P}_E$ reduction map. We have seen that (The paragraph above the proposition 5.2.2)

$$(\pi_E \times \mathrm{id})\{Z_{\mathrm{GL}_r(\mathcal{O}_E/\mathfrak{P}_E^e)\times k_F^\times}(A)\}$$

is a subgroup of $H \times k_F^\times$ where H is a subgroup of $\mathrm{GL}_{n/ef}(k_E)$ whose image under the inclusion map $\mathrm{GL}_{n/ef}(k_E) \hookrightarrow \mathrm{GL}_n(k_F)$ is contained in a proper k_F-parabolic subgroup of $\mathrm{GL}_n(k_F)$. From the result of Paskunas 5.2.2 we get that for every irreducible representation ξ of

$$\mathrm{res}_{Z(\eta_{k_p})}\{U_{k_p} \otimes ((\kappa \otimes \rho) \boxtimes \mathrm{id})\}$$

we can find an irreducible representation $\rho' \not\simeq \rho$ such that ξ occurs in the representation

$$\mathrm{res}_{Z(\eta_{k_p})}\{U_{k_p} \otimes ((\kappa \otimes \rho') \boxtimes \mathrm{id})\}.$$

Hence irreducible sub-representations of

$$\mathrm{ind}_{Z(\eta_{k_p})K_s}^{\mathrm{GL}_{n+1}(\mathcal{O}_F)}\{U_{\eta_{k_p}} \otimes (\lambda \boxtimes \mathrm{id})\}$$

occur as a sub-representation of

$$\mathrm{ind}_{Z(\eta_{k_p})K_s}^{\mathrm{GL}_{n+1}(\mathcal{O}_F)}\{U_{\eta_{k_p}} \otimes ((\kappa \otimes \rho') \boxtimes \mathrm{id})\}.$$

Now the above representation occurs as a sub-representation of

$$\mathrm{ind}_{P_I^0(N_s)}^{\mathrm{GL}_{n+1}(\mathcal{O}_F)}\{(\kappa \otimes \rho') \boxtimes \mathrm{id}\} \simeq \mathrm{ind}_{P_I(N_s)}^{\mathrm{GL}_{n+1}(\mathcal{O}_F)}(\tau' \boxtimes \mathrm{id})\}$$

where τ' is given by

$$\mathrm{ind}_{J^0}^{\mathrm{GL}_n(\mathcal{O}_F)}(\kappa \otimes \rho' \boxtimes \mathrm{id}).$$

Any irreducible representation γ of τ' occurs in an irreducible smooth representation π of $\mathrm{GL}_n(F)$. If ρ' is cuspidal then $\kappa \otimes \rho'$ is contained in the representation γ and hence is contained in π which gives that π is cuspidal but it is not isomorphic to an unramified twist of σ. Now if ρ' is not cuspidal π is not cuspidal. Hence in every case π is not inertially equivalent to σ. This shows that irreducible sub-representations of

$$\mathrm{ind}_{P_I(N_s)}^{\mathrm{GL}_{n+1}(\mathcal{O}_F)}(\gamma \boxtimes \mathrm{id})$$

are atypical. This shows the lemma. □

Theorem 5.3.3. *Let Γ be a typical representation for the component*

$$s = [M_I, \sigma \boxtimes \chi]$$

then Γ is isomorphic to the representation

$$\mathrm{ind}_{J_s}^{\mathrm{GL}_{n+1}(\mathcal{O}_F)}(\lambda_s)$$

where (J_s, λ_s) *is the Bushnell-Kutzko semi-simple type for the component* s*. If* P *is a parabolic subgroup containing* M_I *as a Levi-factor then* Γ *occurs with a multiplicity one in the representation*

$$\mathrm{res}_{\mathrm{GL}_{n+1}(\mathcal{O}_F)}\, i_P^{\mathrm{GL}_{n+1}(F)}(\sigma \boxtimes \chi).$$

Proof. The representation $\mathrm{ind}_{J_s}^{\mathrm{GL}_{n+1}(\mathcal{O}_F)}(\lambda_s)$ is irreducible since the intertwining of this representation is bounded by the group W_s where W_s is the set of representatives for $N_G(s)/M_I$. We can see that in our case W_s is trivial. We refer to [BK98][Lemma 11.5] for these results. Hence the uniqueness of the typical representation. The multiplicity follows from the results 2.2.4, 5.3.1 and 5.3.2. $\square$

Chapter 6

The inertial class with Levi-subgroup of the type (2,2)

6.1 Preliminary elimination of atypical representations

We denote by I the partition $(2,2)$. Let σ_1 and σ_2 be two cuspidal representations of $\mathrm{GL}_2(F)$. We denote by P, M and U the standard parabolic subgroup, the standard Levi-subgroup and the unipotent radical of P corresponding to the partition I. We denote by $\bar{P}$ and $\bar{U}$ the opposite parabolic subgroup of P with respect to the Levi-subgroup M and unipotent radical of $\bar{P}$. In this chapter we are interested in the classification of typical representations for the inertial class $[M, \sigma_1 \boxtimes \sigma_2]$. We denote by π the canonical quotient map

$$\pi : P(\mathcal{O}_F) \to M(\mathcal{O}_F)$$

For any positive integer r we denote by $P(r)$ the inverse image of $P(\mathcal{O}_F/\mathfrak{P}_F^r)$ under the mod-$\mathfrak{P}_F^r$ reduction of $\mathrm{GL}_4(\mathcal{O}_F)$. Let τ_1 and τ_2 be $\mathrm{GL}_2(\mathcal{O}_F)$-typical representations occurring in σ_1 and σ_2 respectively. From the lemma 2.2.4 we get that the representation

$$\mathrm{ind}_{P\cap\mathrm{GL}_4(\mathcal{O}_F)}^{\mathrm{GL}_4(\mathcal{O}_F)}(\tau_1 \boxtimes \tau_2)$$

has a complement say Γ in

$$\mathrm{res}_{\mathrm{GL}_4(\mathcal{O}_F)}\, i_P^{\mathrm{GL}_4(F)}(\sigma_1 \boxtimes \sigma_2)$$

such that $\mathrm{GL}_4(\mathcal{O}_F)$-irreducible sub-representations of Γ are atypical.

Let $[\mathfrak{A}_1, n_1, 0, \beta_1]$ and $[\mathfrak{A}_2, n_2, 0, \beta_2]$ be two simple strata defining simple types (J_1, λ_1) and (J_2, λ_2) contained in σ_1 and σ_2 respectively. We may and do assume that $\mathfrak{A}_1$ and $\mathfrak{A}_2$ are defined by lattice chains $\mathcal{L}_1$ and $\mathcal{L}_2$ such that $\mathcal{L}_1(0) = \mathcal{L}_2(0) = \mathcal{O}_F \oplus \mathcal{O}_F$. We deduce that $\mathfrak{A}_1$ and $\mathfrak{A}_2$ are contained in $M_2(\mathcal{O}_F)$.

The representation λ_i restricted to $U^{[n_i/2]+1}(\mathfrak{A}_i)$ is isomorphic to a direct sum $\oplus^{p^{k_i}} \psi_{\beta_i}$ (for the definition of ψ_β we refer to [BK93][1.1.6]) and hence λ_i is trivial on $U^{n_i+1}(\mathfrak{A}_i)$ for $i \in \{1,2\}$. If $e(\mathfrak{A}_i) = 2$ then

$$K_2([(n_i+1)/2]+1) \subset U^{n_i+1}(\mathfrak{A}_i) \subset K_2([(n_i+1)/2]).$$

Notation 6.1. *We denote by N_{λ_i} the positive integer $[(n_i+1)/2]$ if $e_i = 2$ and n_i if $e_i = 1$. Let N be the positive integer* $\max\{N_{\lambda_1}, N_{\lambda_2}\}$.

The representation $\tau_1 \boxtimes \tau_2$ contains $M \cap K_4(N+1)$ in its kernel. We extend the representation $\tau_1 \boxtimes \tau_2$ of $M(\mathcal{O}_F \,/\, \mathfrak{P}_F{}^r)$ to a representation of $P(r)$ for $r \geq N+1$ via the inflation map

$$\pi_r : P(r) \to P(\mathcal{O}_F \,/\, \mathfrak{P}_F{}^r) \to M(\mathcal{O}_F \,/\, \mathfrak{P}_F{}^r).$$

Note that $P(r) \cap U$ and $P(r) \cap \bar{U}$ are contained in the kernel of this extension. The groups $P(r)$ for $r \geq N+1$ and $\tau = \tau_1 \boxtimes \tau_2$ satisfy the hypothesis for the lemma 2.2.5 hence we obtain

$$\mathrm{ind}_{P \cap \mathrm{GL}_4(\mathcal{O}_F)}^{\mathrm{GL}_4(\mathcal{O}_F)}(\tau_1 \boxtimes \tau_2) = \bigcup_{r \geq N+1} \mathrm{ind}_{P(r)}^{\mathrm{GL}_4(\mathcal{O}_F)}(\tau_1 \boxtimes \tau_2).$$

Hence to classify typical representations we need to examine the typical representations occurring as irreducible sub-representations of

$$\mathrm{ind}_{P(r)}^{\mathrm{GL}_4(\mathcal{O}_F)}(\tau_1 \boxtimes \tau_2)$$

for all $r \geq N+1$. As we did in the previous chapter, it is convenient to work with a smaller subgroup $P^0(r)$. Let r be a positive integer greater than N then $P^0(r) = K_4(r)\pi^{-1}(J_1 \times J_2)$ (We consider $J_1 \times J_2$ as a subgroup of $M(\mathcal{O}_F)$). The group $P^0(r)$ satisfies Iwahori decomposition with respect to P and M. Observe that $P^0(r) \cap U = P(r) \cap U$ and $P^0(r) \cap \bar{U} = P(r) \cap \bar{U}$ and since $r \geq N+1$ we get that $P^0(r) \cap M = J_1^0 \times J_2^0$. Now we apply lemma 2.2.6 for $J_1 = P(r)$ and $J_2 = P^0(r)$ and $\lambda = \lambda_1 \boxtimes \lambda_2$ and obtain an isomorphism

$$\mathrm{ind}_{P(r)}^{\mathrm{GL}_4(\mathcal{O}_F)}(\tau_1 \boxtimes \tau_2) \simeq \mathrm{ind}_{P^0(r)}^{\mathrm{GL}_4(\mathcal{O}_F)}(\lambda_1 \boxtimes \lambda_2).$$

We will need the decomposition of the representation

$$\mathrm{ind}_{P^0(r+1)}^{P^0(r)}(\mathrm{id})$$

for the proof of the theorem 6.1.2. We will prove the theorem by induction on r and the decomposition of the above representation is crucial. Let r be a positive integer greater than or equal to $N+1$. Let $K_I(r)$ be the group

$$K_4(r)\pi^{-1}(K_2(N+1) \times K_2(N+1)).$$

We note that $K_I(r)P^0(r+1) = P^0(r)$. It follows from Mackey decomposition that

$$\mathrm{res}_{K_I(r)}\, \mathrm{ind}_{P^0(r+1)}^{P^0(r)}(\mathrm{id}) \simeq \mathrm{ind}_{K_I(r+1)}^{K_I(r)}(\mathrm{id}).$$

Lemma 6.1.1. *The group $K_I(r)$ is a normal subgroup of $P_I^0(r)$ and $K_I(r+1)$ is a normal subgroup of $K_I(r)$.*

Proof. By definition of the groups $K_I(r)$ we have $K_I(r) \cap U = P^0(r) \cap U$ and $K_I(r) \cap \bar{U} = P^0(r) \cap \bar{U}$. To show the normality of $K_I(r)$ in $P^0(r)$ it is enough to verify that $P^0(r) \cap M$ normalizes the group $K_I(r)$. $P^0(r) \cap M$ normalizes the group $K_I(r) \cap U = U(\mathcal{O}_F)$ and $K_I(r) \cap \bar{U} = \bar{U}(\mathfrak{P}_F^r)$. The group $K_I(r) \cap M$ is a normal subgroup of $P^0(r) \cap M$ and hence $P^0(r) \cap M$ normalizes $K_I(r) \cap M$. This shows the first part of the lemma.

Since $K_I(r) \cap P = K_I(r+1) \cap P$, we have to check that $K_I(r) \cap \bar{U}$ normalizes the group $K_I(r+1)$. We note that $\bar{U}$ is abelian hence we have to check that the conjugations $u^- j (u^-)^{-1}$ and $u^- u^+ (u^-)^{-1}$ belong to the group $K_I(r+1)$ for all $u^- \in K_I(r) \cap \bar{U}_I$, $j \in K_I(r+1) \cap M_I = K_I(r) \cap M_I$ and $u^- \in K_I(r+1) \cap U_I = U_I(\mathcal{O}_F)$. Let us begin with the element $u^- j (u^-)^{-1}$. We have $u^- j (u^-)^- = j\{j^{-1} u^- j (u^-)^{-1}\}$. Let

$$j = \begin{pmatrix} J_1 & 0 \\ 0 & J_2 \end{pmatrix} \quad u^- = \begin{pmatrix} 1_2 & 0 \\ U^- & 1 \end{pmatrix}$$

be the block diagonal form of j and u^-; $J_1 \in K_2(N+1), J_2 \in K_2(N+1)$ and $U^- \in \varpi_F^r M_{2\times 2}(\mathcal{O}_F)$. The element $j^{-1} u^- j (u^-)^{-1}$ is of the form

$$\begin{pmatrix} 1_2 & 0 \\ J_2^{-1} U^- J_1 - U^- & 1_2 \end{pmatrix}$$

We note that the matrix $J_1^{-1} U^- J_1 - U^-$ belongs to $\varpi_F^{r+1} M_{2\times 2}(\mathcal{O}_F)$. This shows that $j^{-1} u^- j (u^-)^{-1} \in K_I(r+1) \cap \bar{U}_I$. Hence the element $u^- j (u^-)^-$ which can be rewritten as $j\{j^{-1} u^- j (u^-)^{-1}\}$ belongs to $K_I(r+1)$.

We now consider the conjugation $u^- u^+ (u^-)^{-1}$. We write u^+ in its block matrix form as

$$\begin{pmatrix} 1_2 & U^+ \\ 0 & 1_2 \end{pmatrix}$$

where $U^+ \in M_{2\times 2}(\mathcal{O}_F)$. Now the conjugation $u^- u^+ (u^-)^{-1}$ in its block matrix from is as follows

$$\begin{pmatrix} 1_2 - U^+ U^- & U^+ \\ -U^- U^+ U^- & U^- U^+ + 1_2 \end{pmatrix}.$$

Since $U^- U^+ U^- \in \varpi^{2r} M_{2\times 2}(\mathcal{O}_F)$ and $2r \geq r+1$, we conclude that $u^- u^+ (u^-)^{-1}$ belongs to $K_I(r+1)$. This ends the proof of this lemma. $\square$

From the above lemma and Iwahori decomposition for the group $K_I(r)$ we get that the inclusion $K_I(r) \cap \bar{U}$ in $K_I(r)$ induces an isomorphism of the quotient $K_I(r)/K_I(r+1)$ with

$$(K_I(r) \cap \bar{U})/(K_I(r+1) \cap \bar{U}). \tag{6.1}$$

The representation $\mathrm{ind}_{K_I(r+1)}^{K_I(r)}(\mathrm{id})$ splits as a direct sum of characters of $K_I(r)$ which are trivial on $K_I(r+1)$. We denote these characters by η_k for $1 \leq k \leq t$. The group $P^0(r)$ acts on these characters and let $Z(\eta_k)$ be the $P^0(r)$-stabilizer of the character η_k. We note that the trivial character id occurs with multiplicity one. From Clifford theory we get that

$$\mathrm{ind}_{P^0(r+1)}^{P^0(r)}(\mathrm{id}) = \mathrm{id} \oplus \bigoplus_{\eta_{n_k} \neq \mathrm{id}} \mathrm{ind}_{Z(\eta_{n_k})}^{P^0(r)}(U_{\eta_{n_k}}) \tag{6.2}$$

where η_{n_k} is a representative for the $P^0(r)$-orbit and $U_{\eta_{n_k}}$ is an irreducible representation of $Z(\eta_{n_k})$. Note that $Z(\eta_{n_k}) = (Z(\eta_{n_k}) \cap M)K_I(r)$.

The next step is to bound the group $Z(\eta_{n_k}) \cap M$ for some $\eta_{n_k} \neq \mathrm{id}$. Let u^- be an element of the group $K_I(r) \cap \bar{U}_I$. We represent the element u^- in its block form as

$$\begin{pmatrix} 1_2 & 0 \\ U^- & 1_2 \end{pmatrix}.$$

The map $u^- \mapsto \varpi_F^{-r}U^-$ gives us an isomorphism of $K_I(r) \cap \bar{U}_I$ with $M_2(\mathcal{O}_F)$. Further the map $u^- \mapsto \overline{\varpi_F^{-r}U^-}$ (we denote by $\overline{U}$ the class of $U \in M_2(\mathcal{O}_F)$ in the quotient $M_2(\mathcal{O}_F)/\varpi_F M_2(\mathcal{O}_F)$) gives an isomorphism of the quotient $(K_I(r) \cap \bar{U}_I)/(K_I(r+1) \cap \bar{U}_I)$ with $M_{2\times 2}(k_F)$. This is $M(\mathcal{O}_F)$-equivariant. We also have an $M(\mathcal{O}_F)$-equivariant isomorphism between the character group of $M_{2\times 2}(k_F)$ and $M_{2\times 2}(k_F)$ (see 3.0.14). We finally obtain an $M(\mathcal{O}_F)$-equivariant isomorphism

$$K_I(r)\widehat{/K_I(}r+1) \simeq M_{2\times 2}(k_F). \tag{6.3}$$

Note that the group $M(\mathcal{O}_F)$ acts through its quotient $M(k_F)$.

In order to calculate $Z(\eta_{n_k}) \cap M$ for some $\eta_{n_k} \neq \mathrm{id}$, we can as well calculate $Z_{J_1 \times J_2}(m)$ for some non-zero matrix m in $M_2(k_F)$. It will be useful to first recall the $Z_{M(k_F)}(m)$ for $m \neq 0$. We have the following possibilities

1. If m is a full rank matrix then $Z_{M(k_F)}(m) = \{(g, mgm^{-1}) \,|g \in \mathrm{GL}_2(k_F)\}$.
2. If m is not a full rank matrix then $Z_{M(k_F)}(m) = \{(g_1, g_2) \,|g_2 m = m g_1 \}$. Since m is non-zero and has a kernel, we can see that g_1 fixes the kernel hence the first projection of $Z_{M(k_F)}(m)$ is contained in a proper parabolic subgroup of $\mathrm{GL}_2(k_F)$. The conclusion in this case is symmetric for the second projection as well.

We denote by λ_s the representation $\lambda_1 \boxtimes \lambda_2$ of $J_1 \times J_2$. The representation λ_s occurs with multiplicity one in $\mathrm{ind}_{P^0(r)}^{P^0(N+1)}(\lambda_s)$. We denote by $U_r^0(\lambda_s)$ the complement of λ_s in $\mathrm{ind}_{P^0(r)}^{P^0(N+1)}(\lambda_s)$. Let $U_r(\lambda_s)$ be the representation

$$\mathrm{ind}_{P^0(N+1)}^{\mathrm{GL}_4(\mathcal{O}_F)}(U_r^0(\lambda_s)).$$

Theorem 6.1.2. *Let $\#k_F > 3$. The $\mathrm{GL}_4(\mathcal{O}_F)$-irreducible sub-representations of $U_r(\lambda_s)$ are atypical for all $r \geq N+1$.*

Proof. We prove this theorem by induction on the integer $r \geq N+1$. The theorem is vacuously true for $r = N+1$ since $U_r(\lambda_s) = 0$. We suppose that the theorem is true for some positive integer r we will prove the same holds for $r+1$. We first note that

$$\mathrm{ind}_{P^0(r+1)}^{P^0(N+1)}(\lambda_s) \simeq \mathrm{ind}_{P^0(r)}^{P^0(N+1)}\{\mathrm{ind}_{P^0(r+1)}^{P^0(r)}(\mathrm{id}) \otimes \lambda_s\}.$$

Now using the decomposition 6.2 we get that

$$\mathrm{ind}_{P^0(r+1)}^{P^0(N+1)}(\lambda_s) \simeq \mathrm{ind}_{P^0(r)}^{P^0(N+1)}(\lambda_s) \bigoplus_{\eta_{n_k} \neq \mathrm{id}} \mathrm{ind}_{Z(\eta_{n_k})}^{P^0(N+1)}(\lambda_s \otimes U_{\eta_{n_k}}).$$

From the definition of $U_r^0(\lambda_s)$ we get that

$$U_{r+1}^0(\lambda_s) \simeq U_r^0(\lambda_s) \oplus \bigoplus_{\eta_{n_k} \neq \mathrm{id}} \mathrm{ind}_{Z(\eta_{n_k})}^{P^0(N+1)}(\lambda_s \otimes U_{\eta_{n_k}}). \tag{6.4}$$

Now applying the induction functor to the maximal compact subgroup $\mathrm{GL}_4(\mathcal{O}_F)$ we have

$$U_{r+1}(\lambda_s) \simeq U_r(\lambda_s) \oplus \bigoplus_{\eta_{n_k} \neq \mathrm{id}} \mathrm{ind}_{Z(\eta_{n_k})}^{\mathrm{GL}_n(\mathcal{O}_F)}(\lambda_s \otimes U_{\eta_{n_k}}). \tag{6.5}$$

Let $\eta_{n_k} \neq \mathrm{id}$. We will show that the irreducible sub-representations of

$$\mathrm{ind}_{Z(\eta_{n_k})}^{\mathrm{GL}_n(\mathcal{O}_F)}(\lambda_s \otimes U_{\eta_{n_k}}) \tag{6.6}$$

are atypical for the component $[M, \sigma_1 \boxtimes \sigma_2]$. We choose to treat this case by case depending on the different classes of supercuspidal representations σ_1 and σ_2.

Case 1: We will first consider the case where both σ_1 and σ_2 are twists of level-zero cuspidal representations. From our assumptions $\lambda_i \simeq \chi_i \otimes \lambda_i'$ where χ_i is a character of $F^\times$ and λ_i' is the inflation of a cuspidal representation of $\mathrm{GL}_2(k_F)$ for $i \in \{1, 2\}$. Let m be the non-zero matrix associated to the non-trivial character η_{n_k}. We observe that $Z(\eta_{n_k}) \cap M$ is equal to $Z_{\mathrm{GL}_2(\mathcal{O}_F)\times\mathrm{GL}_2(\mathcal{O}_F)}(m)$. Hence mod $\mathfrak{P}_F$ reduction of $Z(\eta_{n_k}) \cap M$ satisfies one of the properties listed in (6.3). We know from the results of chapter three (see 3.0.16, 3.0.17) that any irreducible sub-representation of

$$\mathrm{res}_{Z(\eta_{n_k})}(\chi_1\lambda_1' \boxtimes \chi_2\lambda_2')$$

occur in

$$\mathrm{res}_{Z(\eta_{n_k})}(\chi_1\lambda_1'' \boxtimes \chi_2\lambda_2'')$$

where $\lambda_1''\boxtimes\lambda_2''$ is the inflation of a non-cuspidal representation of $M(k_F)$. Hence we deduce that the irreducible sub-representations of

$$\mathrm{ind}_{Z(\eta_{n_k})}^{\mathrm{GL}_n(\mathcal{O}_F)}(\lambda_s \otimes U_{\eta_{n_k}})$$

occurs in the representation

$$\mathrm{ind}_{Z(\eta_{n_k})}^{\mathrm{GL}_2(\mathcal{O}_F)}\{(\chi_1\lambda_1'' \boxtimes \chi_2\lambda_2'') \otimes U_{\eta_{n_k}}\}$$

which occurs in the representation

$$\mathrm{ind}_{P\cap \mathrm{GL}_4(\mathcal{O}_F)}^{\mathrm{GL}_4(\mathcal{O}_F)}(\chi_1\lambda_1'' \boxtimes \chi_2\lambda_2'').$$

Since $\lambda_1'' \boxtimes \lambda_2''$ is the inflation of a non-cuspidal representation of $M(k_F)$, the above representation occurs in

$$\mathrm{res}_{\mathrm{GL}_4(\mathcal{O}_F)}\, i_{P'}^{\mathrm{GL}_4(F)}(\pi)$$

where P' is a parabolic subgroup contained properly in P. This shows that the irreducible sub-representations of (6.6) are atypical.

Case 2: Let (J_1, λ_1) be defined by a simple strata $[\mathfrak{A}, n_1, 0, \beta_1]$ such that $[E_1 := F[\beta_1] : F] > 1$ and (J_2, λ_2) be such that $J_2 = \mathrm{GL}_2(\mathcal{O}_F)$ and λ_2 is $\chi \otimes \lambda_2'$ where χ is a character of $\mathrm{GL}_2(F)$ and λ_2' is the inflation of a cuspidal representation of $\mathrm{GL}_2(k_F)$. Let m be the matrix in $M_{2\times 2}(k_F)$ associated to the non-trivial character η_{n_k} in the isomorphism (6.3). We first consider the easier case when m is a non-zero matrix with rank one. We note that the second projection of $Z_{M(k_F)}(m)$ is contained in a proper parabolic subgroup say B of $\mathrm{GL}_2(k_F)$. Hence by lemma 3.0.16 we get that for any irreducible sub-representation ξ of $\mathrm{res}_B\,\lambda_2'$ there exists a $\mathrm{GL}_2(k_F)$-irreducible non-cuspidal representation λ_3 such that ξ occurs as a sub-representation of $\mathrm{res}_B\,\lambda_3$ and $\lambda_2' \not\simeq \lambda_3$. This shows that any irreducible sub-representation of

$$\mathrm{ind}_{Z(\eta_{n_k})}^{\mathrm{GL}_n(\mathcal{O}_F)}(\lambda_s \otimes U_{\eta_{n_k}})$$

is contained in

$$\mathrm{ind}_{Z(\eta_{n_k})}^{\mathrm{GL}_n(\mathcal{O}_F)}((\lambda_1 \boxtimes \chi\lambda_3') \otimes U_{\eta_{n_k}}).$$

for some irreducible representation λ_3' of $\mathrm{GL}_2(\mathcal{O}_F)$ obtained by inflating an irreducible representation λ_3 of $\mathrm{GL}_2(k_F)$. Hence for every irreducible sub-representation say γ of

$$\mathrm{ind}_{Z(\eta_{n_k})}^{\mathrm{GL}_4(\mathcal{O}_F)}(\lambda_s \boxtimes U_{\eta_{n_k}})$$

there exists λ_3' (obtained by inflating an irreducible non-cuspidal representation of $\mathrm{GL}_2(k_F)$) such that γ occurs as sub-representation of

$$\mathrm{ind}_{P^0(m+1)}^{\mathrm{GL}_4(\mathcal{O}_F)}(\lambda_1 \boxtimes \chi\lambda_3').$$

The above representation is contained in

$$\mathrm{ind}_{P\cap \mathrm{GL}_4(\mathcal{O}_F)}^{\mathrm{GL}_4(\mathcal{O}_F)}(\sigma_1 \boxtimes \sigma_2')$$

where σ_2' is an irreducible smooth representation of $\mathrm{GL}_2(F)$ which contains the type $(\mathrm{GL}_2(\mathcal{O}_F), \chi\lambda_3')$. Since λ_3' is the inflation of a non-cuspidal representation of $\mathrm{GL}_2(k_F)$, σ_2' is not a cuspidal representation hence the irreducible sub-representations of

$$\mathrm{ind}_{Z(\eta_{n_k})}^{\mathrm{GL}_n(\mathcal{O}_F)}(\lambda_s \boxtimes U_{\eta_{n_k}})$$

are atypical.

This paragraph concerns the case where m is a matrix of full rank. We begin with the observation that $Z(\eta_{n_k}) \cap M = Z_{J_1\times J_2}(\eta_{n_k})$. Let $[E_1 = F[\beta_1] : F] > 1$. The group J_1 contains a normal subgroup J_1^1 such that $J_1 = \mathcal{O}_{E_1}^\times J_1^1$ and $J_1/J_1^1 \simeq \mathcal{O}_{E_1}^\times/U^1(E_1)$ and J_1^1 is a pro-p subgroup (see [BK93, Chapter 3, 3.1.14]). Now we have to understand $Z_{\mathcal{O}_{E_1}^\times J_1^1\times \mathrm{GL}_2(\mathcal{O}_F)}(m)$. The action of the group $(\mathcal{O}_{E_1}^\times J_1^1)\times \mathrm{GL}_2(\mathcal{O}_F)$ factors through its quotient

$$(U^0(E_1)J_1^1K_2(1)/K_2(1)) \times \mathrm{GL}_2(k_F).$$

We now have two different possibilities: $e_1 = 1$ and $e_1 = 2$.

If $e_1 = 1$ then we have $U^0(E_1)J_1^1K_2(1)/K_2(1) = U^0(E_1)/U^{e_1}(E_1) = k_{E_1}^\times$ where k_{E_1} is a quadratic extension of k_F. Now it is clear that

$$Z_{U^0(E_1)/U^{e_1}(E_1)\times \mathrm{GL}_2(k_F)}(m) \cap (\{\mathrm{id}\}\times U) = \{\mathrm{id}\}$$

where U is the unipotent radical of any Borel subgroup of $\mathrm{GL}_2(k_F)$.

If $e_1 = 2$ the group $U^0(E_1)J_1^1K_2(1)/K_2(1)$ is $k_F^\times X$ where X is a p-group. Now the second projection of $Z_{k_F^\times X\times \mathrm{GL}_2(k_F)}(m)$ is contained in the product of the center $k_F^\times$ and a p-group. Hence we conclude that the mod $\mathfrak{P}_F$-reduction of the image of the second projection of $Z_{J_1\times J_2}(m)$ is contained in a Borel subgroup say B of $\mathrm{GL}_2(k_F)$. If $\bar{B}$ is the opposite Borel subgroup then its its unipotent radical U satisfies the property that

$$Z_{k_F^\times X\times \mathrm{GL}_2(k_F)}(m) \cap (\{\mathrm{id}\}\times U) = \{\mathrm{id}\}.$$

Let H be a subgroup of $\mathrm{GL}_2(k_F)$ such that $H\cap U = \{\mathrm{id}\}$ and σ be a cuspidal representation of $\mathrm{GL}_2(k_F)$. For any irreducible sub-representation ξ of $\mathrm{res}_H(\sigma)$ we can find an irreducible non-cuspidal representation σ' of $\mathrm{GL}_2(k_F)$ such that $\mathrm{Hom}_H(\xi, \sigma') \neq 0$ and $\sigma' \not\simeq \sigma$. This is because Mackey decomposition shows that

$$\mathrm{Hom}_U(\mathrm{ind}_H^{\mathrm{GL}_2(k_F)}(\xi), \mathrm{id}) \neq 0$$

and hence $\mathrm{ind}_H^{\mathrm{GL}_2(k_F)}(\xi)$ cannot be a sum of cuspidal representations. This shows that for any irreducible sub-representation γ of

$$\mathrm{ind}_{Z(\eta_{n_k})}^{\mathrm{GL}_n(\mathcal{O}_F)}(\lambda_s \otimes U_{\eta_{n_k}})$$

there exists an irreducible representation λ_3 of $\mathrm{GL}_2(\mathcal{O}_F)$ obtained by inflating a non-cuspidal representation of $\mathrm{GL}_2(k_F)$ such that γ occurs in

$$\mathrm{ind}_{Z(\eta_{n_k})}^{\mathrm{GL}_n(\mathcal{O}_F)}\{(\lambda_1 \boxtimes \chi\lambda_3) \boxtimes U_{\eta_{n_k}}).$$

The above representation occurs in the representation

$$\mathrm{ind}_{P^0(m+1)}^{\mathrm{GL}_4(\mathcal{O}_F)}\{(\lambda_1 \boxtimes \chi\lambda_3) \boxtimes U_{\eta_{n_k}})$$

and we have

$$\mathrm{ind}_{P^0(m+1)}^{\mathrm{GL}_4(\mathcal{O}_F)}\{(\lambda_1 \boxtimes \chi\lambda_3) \boxtimes U_{\eta_{n_k}}) \subset \mathrm{ind}_{P\cap \mathrm{GL}_4(\mathcal{O}_F)}^{\mathrm{GL}_4(\mathcal{O}_F)}(\sigma_1 \boxtimes \sigma_2')$$

the representation σ_2' is an irreducible smooth representation containing the type $(\mathrm{GL}_2(\mathcal{O}_F), \chi\lambda_3)$. hence γ is not typical representation. This shows that irreducible sub-representations of (6.6) are atypical.

Case 3: We now consider the case where both (J_1, λ_1) and (J_2, λ_2) are defined by simple strata $[\mathfrak{A}_1, n_1, 0, \beta_1]$ and $[\mathfrak{A}_2, n_2, 0, \beta_2]$ respectively such that $[E_i = F[\beta_i] : F] > 1$ where $i \in \{1, 2\}$. We have to look at the stabilizer $Z(\eta_{n_k}) \cap M = Z_{J_1 \times J_2}(m)$. We consider the possibilities $(e_1, e_2) = (1, 1)$; $(e_1, e_2) = (2, 1)$ and $(e_1, e_2) = (2, 2)$, the other case is similar.

Case 3.1: Here $e_1 = 1$ and $e_2 = 1$, $J_1^1 \times J_2^1$ is contained in the group $K_2(1) \times K_2(1)$. The group $J_1 \times J_2$ acts through its mod-$\mathfrak{P}_F$ reduction hence it acts through its quotient $k_{E_1}^\times \times k_{E_2}^\times$. We are reduced to bound the group $Z_{k_{E_1}^\times \times k_{E_2}^\times}(m)$. Let m be a matrix of rank one and $(a, b) \in Z_{k_{E_1}^\times \times k_{E_2}^\times}(m)$. Now $am = mb$ and let v be a vector in k_F^2 which is contained in the kernel of m. Now $mb(v) = 0$ implies that $b(v)$ is in the kernel of m. This shows that b has eigen-values in k_F hence we must have $b \in k_F^\times$, similarly $a \in k_F^\times$ and $am = mb$ implies that $a = b$ (since at least one of the entries of m is non-zero). If m is a matrix of full rank then $a = mbm^{-1}$ implies that $a \in mk_{E_2}^{-1}m^\times \cap k_{E_1}^\times$. Now $mk_{E_2}m^{-1} \cap k_{E_1}$ is a sub-field of k_{E_1} and there are two possibilities: either $mk_{E_2}m^{-1} \cap k_{E_1}$ is a proper sub-field or $mk_{E_2}m^{-1} = k_{E_1}$. The first one would imply that $a, b \in k_F^\times$ and $a = b$. We conclude that $Z_{k_{E_1}^\times \times k_{E_2}^\times}(m)$ has the form

$$\{(a, a) \mid a \in k_F^\times\}$$

or there exist a field isomorphism θ of k_{E_1} onto k_{E_2} such that

$$Z_{k_{E_1}^\times \times k_{E_2}^\times}(m) = \{(a, \theta(a)) \mid a \in k_{E_1}^\times\}.$$

In every possibility the mod $\mathfrak{P}_F$ reduction of $Z_{J_1^0\times J_2^0}(m)$ is a subgroup of $\{(e,\theta(e) \mid e\in k_E^\times\}$ for some θ. Let $\chi_1=(\theta\circ\chi_2)^{-1}$ and χ_2 be two non-trivial characters of J_1 and J_2 which are trivial on J_1^1 and J_2^1 respectively. Let σ_1' and σ_2' be two supercuspidal representations containing Bushnell-Kutzko types $(J_1,\lambda_1'=\lambda_1\otimes\chi_1)$ and $(J_2,\lambda_2'=\lambda_2\otimes\chi_2)$ respectively. If $(M,\sigma_1\boxtimes\sigma_2)$ and $(M,\sigma_1'\boxtimes\sigma_2')$ are inertially equivalent for all χ_2 then we must have intertwining between (J_1,λ_1) and $(J_2,\lambda_2\otimes\chi_2)$ (because (J_1,λ_1) and $(J_1,\lambda_1\otimes\chi_1)$ cannot intertwine see [BK93, Chapter 5, Theorem 5.5.2(3)]) for any non-trivial character χ_2 of J_2^0/J_2^1. Now we use the intertwining implies conjugacy theorem [BK93, Chapter 6, 6.2.4] to get a $g\in \mathrm{GL}_2(F)$ such that $J_2=gJ_1g^{-1}$ and $\lambda_1^g\simeq\lambda_2\otimes\chi_2$. Since conjugacy is an equivalence relation we get that $(J_2,\lambda_2\otimes\chi_2)$ are all conjugate for all non-trivial characters χ_2 of J_2/J_2^1. There exist two distinct non-trivial characters of $k_{E_2}^\times$ (since the cardinality of $k_{E_2}^\times$ is at least 3) hence we get a contradiction to the assumption that $(M,\sigma_1\boxtimes\sigma_2)$ and $(M,\sigma_1'\boxtimes\sigma_2')$ are inertially equivalent. Moreover by definition we have

$$\mathrm{res}_{Z_{J_1\times J_2}(m)}(\lambda_1\boxtimes\lambda_2)\simeq\mathrm{res}_{Z_{J_1\times J_2}(m)}(\lambda_1'\boxtimes\lambda_2').$$

With this observation we have

$$\mathrm{ind}_{Z(\eta_{n_k})}^{\mathrm{GL}_n(\mathcal{O}_F)}((\lambda_1\boxtimes\lambda_2)\boxtimes U_{\eta_{n_k}})\simeq\mathrm{ind}_{Z(\eta_{n_k})}^{\mathrm{GL}_n(\mathcal{O}_F)}((\lambda_1'\boxtimes\lambda_2')\boxtimes U_{\eta_{n_k}}).$$

Hence the irreducible sub-representations of

$$\mathrm{ind}_{Z(\eta_{n_k})}^{\mathrm{GL}_n(\mathcal{O}_F)}(\lambda_s\boxtimes U_{\eta_{n_k}})$$

are atypical.

Case 3.2: Let us consider the case where $e_1=1$ and $e_2=2$. The group $J_1\times J_2$ now acts via the quotient $k_{E_1}^\times\times U^0(E_2)J_2^1K_2(1)/K_2(1)$. Let m be the matrix associated to the character η_{n_k}. If m has rank one then every element of the first projection of $Z_{k_{E_1}^\times\times U^0(E_2)J_2^1K_2(1)/K_2(1)}(m)$ is contained in $k_F^\times$. If m is a full rank matrix then for all $(a,b)\in Z_{k_{E_1}^\times\times U^0(E_2)J_2^1K_2(1)/K_2(1)}(m)$ we have $a=mbm^{-1}$. Note that $U^0(E_2)J_2^1K_2(1)/K_2(1)$ is a product $k_F^\times X$ where X is a p-group. This shows that the first projection is contained in $k_F^\times$. In each case the first projection of $Z_{k_{E_1}^\times\times U^0(E_2)J_2^1K_2(1)/K_2(1)}(m)$ is contained in $k_F^\times$. Let χ be a character of $k_{E_1}^\times$ which is trivial on $k_F^\times$. Such a character exists since the cardinality of $k_{E_1}^\times/k_F^\times$ is $q+1\geq 3$. Let σ_1' be a cuspidal representation of $\mathrm{GL}_2(F)$ containing the type $(J_1,\lambda_1\otimes\chi)$. We note that σ_1 and σ_2 are not inertial twist of each other (see [BK93, Chapter 5, Theorem 5.5.2(3)]). Hence $(M,\sigma_1\boxtimes\sigma_2)$ and $(M,\sigma_1'\boxtimes\sigma_2)$ are not inertially equivalent. Moreover we have

$$\mathrm{res}_{Z_{J_1\times J_2}(m)}(\lambda_1\boxtimes\lambda_2)\simeq\mathrm{res}_{Z_{J_1\times J_2}(m)}(\lambda_1'\boxtimes\lambda_2).$$

With this observation we have

$$\mathrm{ind}_{Z(\eta_{n_k})}^{\mathrm{GL}_n(\mathcal{O}_F)}((\lambda_1\boxtimes\lambda_2)\boxtimes U_{\eta_{n_k}})\simeq\mathrm{ind}_{Z(\eta_{n_k})}^{\mathrm{GL}_n(\mathcal{O}_F)}((\lambda_1'\boxtimes\lambda_2)\boxtimes U_{\eta_{n_k}}).$$

Hence the irreducible sub-representations of

$$\mathrm{ind}_{Z(\eta_{n_k})}^{\mathrm{GL}_n(\mathcal{O}_F)}(\lambda_s \boxtimes U_{\eta_{n_k}})$$

are atypical for $s = [M, \sigma_1 \boxtimes \sigma_2]$.

Case 3.3: We are left with the case where $e_1 = 2$ and $e_2 = 2$. The group $(J_1 \times J_2)$ acts through its quotient

$$U^0(E_1)J_1^1 K_2(1)/K_2(1) \times U^0(E_2)J_2^1 K_2(1)/K_2(1).$$

We write this quotient as $k_F^\times X_1 \times k_F^\times X_2$ where X_1 and X_2 are two p-groups. The group $X_1 \times X_2$ is a p-group and there is a decreasing filtration of $M_2(k_F)$ by sub-spaces $\mathcal{F}^i$ such that $X_1 \times X_2$ acts trivially on $\mathcal{F}^i/\mathcal{F}^{i+1}$ and $\cap_i \mathcal{F} = 0$. Let k be the largest positive integer such that $m \in \mathcal{F}^k$. Let $\bar{m}$ be the image of m in $\mathcal{F}^k/\mathcal{F}^{k+1}$. If $(ax_1, bx_2) \in Z_{k_F^\times X_1, k_F^\times X_2}(m)$ then (ax_1, bx_2) fixes the element $\bar{m}$. The group $k_F^\times \times k_F^\times$ acts on $M_2(k_F)$ by the character ψ given by $\psi(m) = bma^{-1}$. The group $k_F^\times \times k_F^\times$ has cardinality relatively prime to p. Hence we get that the action of $k_F^\times \times k_F^\times$ on $\mathcal{F}^k/\mathcal{F}^{k+1}$ decomposes as a direct sum of isomorphic copies of ψ. From this we conclude that $a = b$.

Let η be a non-trivial character of the group $J_1/J_1^1 = k_F^\times$. Let σ_1' and σ_2' be two cuspidal representations containing the Bushnell-Kutzko types $(J_1, \lambda_1 \otimes \eta)$ and $(J_1, \lambda_1 \otimes \eta^{-1})$. If the pairs $(M, \sigma_1 \boxtimes \sigma_2)$ and $(M, \sigma_1' \boxtimes \sigma_2')$ are inertially equivalent for every η then we have (J_1, λ_1) and $(J_2, \lambda_2 \otimes \eta^{-1})$ must intertwine and hence they should be G-conjugate which implies that $(J_2, \lambda_2 \otimes \eta^{-1})$ are all G conjugate. Now by our assumption that $\#k_F > 3$, we can find two distinct non-trivial characters of $k_F^\times$ which is a contradiction by [BK93, Chapter 5, Theorem 5.5.2(3)]. Hence there is a non-trivial character η of J_1^0/J_1^1 such that $[M, \sigma_1 \boxtimes \sigma_2]$ and $[M, \sigma_1' \boxtimes \sigma_2']$ are distinct inertial classes and

$$\mathrm{res}_{Z_{J_1 \times J_2}(m)}(\lambda_1 \boxtimes \lambda_2) \simeq \mathrm{res}_{Z_{J_1 \times J_2}(m)}(\lambda_1 \eta \boxtimes \lambda_2 \eta^{-1}).$$

With this observation we have

$$\mathrm{ind}_{Z(\eta_{n_k})}^{\mathrm{GL}_n(\mathcal{O}_F)}((\lambda_1 \boxtimes \lambda_2) \boxtimes U_{\eta_{n_k}}) \simeq \mathrm{ind}_{Z(\eta_{n_k})}^{\mathrm{GL}_n(\mathcal{O}_F)}((\lambda_1 \eta \boxtimes \lambda_2 \eta^{-1}) \boxtimes U_{\eta_{n_k}}).$$

Hence the irreducible sub-representations of

$$\mathrm{ind}_{Z(\eta_{n_k})}^{\mathrm{GL}_n(\mathcal{O}_F)}(\lambda_s \boxtimes U_{\eta_{n_k}})$$

are atypical.

Now by using induction on the positive integer r we prove that $U_r(\lambda_s)$ does not contain any atypical representations. $\square$

The previous theorem reduces the problem of classifying typical representations for the component $s = [M, \sigma_1 \boxtimes \sigma_2]$ to classifying typical representations occurring in the representation

$$\mathrm{ind}_{P^0(N+1)}^{\mathrm{GL}_4(\mathcal{O}_F)}(\lambda_s) \tag{6.7}$$

where λ_s is a $J_1 \times J_2$ representation isomorphic to $\lambda_1 \boxtimes \lambda_2$. The representation (6.7) may still contain atypical representations. To examine this we need the Bushnell-Kutzko semi-simple type (J_s, λ_s) for the component s. To write explicitly the structure of the semi-simple type (J_s, λ_s) we need to recall the characteristic polynomial associated to the strata $[\mathfrak{A}, n, 0, \beta]$. For any given strata $[\mathfrak{A}, n, 0, \beta]$ in $M_2(F)$, define g to be $\gcd(n, e)$. The element $\varpi_F^{n/g}\beta^{e/g}$ lies in the ring $\mathfrak{A}$. Since we assume that our hereditary order $\mathfrak{A}$ is defined by a lattice chain $\mathcal{L}$ such that $\mathcal{L}(0) = \mathcal{O}_F \oplus \mathcal{O}_F$, the element $\varpi_F^{n/g}\beta^{e/g}$ belongs to (the maximal hereditary order containing $\mathfrak{A}$) $M_2(\mathcal{O}_F)$. The characteristic polynomial associated to the class $\overline{\varpi_F^{n/g}\beta^{e/g}}$ in $M_2(k_F)$ will be called the characteristic polynomial associated to the strata $[\mathfrak{A}, n, 0, \beta]$.

Let $[\mathfrak{A}_1, n_1, 0, \beta_1]$ and $[\mathfrak{A}_2, n_2, 0, \beta_2]$ be two simple strata defining the maximal simple types (J_1^0, λ_1) and (J_2^0, λ_2) respectively. In our situation the characteristic polynomials associated to the above strata are powers of irreducible polynomials ϕ_1 and ϕ_2 respectively (see [BK93][2.3.11]). The underlying compact group J_s of the semi-simple type depends on the data $n_1/e_1, n_2/e_2$ and ϕ_1 and ϕ_2. We have two possibilities:

1. $n_1/e_1 \neq n_2/e_2$ or $n_1/e_1 = n_2/e_2$ but $\phi_1 \neq \phi_2$
2. $n_1/e_1 = n_2/e_2$ and $\phi_1 = \phi_2$.

In the first case σ_1 and σ_2 are said to be completely distinct. In the second case σ_1 and σ_2 are said to have common approximation. We will classify typical representations in two important cases: the first case when σ_1 and σ_2 have a common approximation of level zero, they are also called homogenous inertial classes and the second case where σ_1 and σ_2 are completely distinct.

6.2 Homogenous inertial classes

In this section we assume that $\#k_F > 3$. So far we have shown that typical representations occur as sub-representations of

$$\mathrm{ind}_{P^0(N+1)}^{\mathrm{GL}_4(\mathcal{O}_F)}(\lambda_1 \boxtimes \lambda_2).$$

This may not (although we believe that this is never the complete classification when σ_1 and σ_2 have common approximation) be the complete classification in the case where σ_1 and σ_2 admit common approximation. In this section we treat the case where $n_1/e_1 = n_2/e_2$ and $\phi_1 = \phi_2$; and σ_1 and σ_2 have level-zero common approximation. This means that the simple characters of σ_1 and σ_2 intertwine and hence they are conjugate. We may as well assume that $\mathfrak{A}_1$ is equal to $\mathfrak{A}_2$, n_1 is equal to n_2, β_1 is equal to β_2. We henceforth assume that σ_1 and σ_2 contain the simple strata $[\mathfrak{A}, n, 0, \alpha]$. Moreover the simple characters defining σ_1 and σ_2 are the same. We denote by $E = F[\alpha]$ and $[E : F] > 1$. We refer to [BK99][Section 4.3] for further details.

Let σ_1 and σ_2 be two supercuspidal representations of $\mathrm{GL}_2(F)$ containing the simple stratum $[\mathfrak{A}, n, 0, \alpha]$. Let (J^0, λ_1) and (J^0, λ_2) be Bushnell-Kutzko type,s associated to $[\mathfrak{A}, n, 0, \alpha]$, contained in σ_1 and σ_2 respectively. We also define a non-negative integer $t = [n/2]$. After twisting by a character χ of $\mathrm{GL}_4(F)$ we may assume that α is minimal in the sense [BK93][1.4.14]. The group J_s in the Bushnell-Kutzko type (J_s, λ_s) for the component $s = [M, \sigma_1 \boxtimes \sigma_2]$ is given by

$$\begin{pmatrix} J^0 & \mathfrak{J}^0 \\ \mathfrak{H}^1 & J^0 \end{pmatrix}.$$

We refer to [BK99][Section 7.2] for this construction. We also refer to the article [Blo06][Corollaire 1] for an exposition. It follows from the minimality of β that

$$\mathfrak{H}^1 = \mathfrak{P}_E + \mathfrak{P}_{\mathfrak{A}}^{t+1} \text{ and } \mathfrak{J}^0 = \mathcal{O}_E + \mathfrak{P}_{\mathfrak{A}}^{n-t}.$$

We refer to [BK93][Definition 3.1.7] for the definition of the lattices $\mathfrak{H}^1$ and $\mathfrak{J}^0$. From the above description the group J_s is of the form

$$J_s = \begin{pmatrix} J^0 & \mathcal{O}_E + \mathfrak{P}_{\mathfrak{A}}^{n-t} \\ \mathfrak{P}_E + \mathfrak{P}_{\mathfrak{A}}^{t+1} & J^0 \end{pmatrix}.$$

The representation $\lambda_1 \boxtimes \lambda_2$ of $J^0 \times J^0$ extends to a representation of J_s such that $J_s \cap U$ and $J_s \cap \bar{U}$ are contained in the kernel of the extension. We denote by λ_s the extension of $\lambda_1 \boxtimes \lambda_2$. The pair (J_s, λ_s) is the Bushnell-Kutzko semi-simple type for the component s.

6.2.1 The complete classification when E is unramified

We first understand the case when E is an unramified extension of F. In particular $\mathfrak{A} = M_2(\mathcal{O}_F)$.

Recall that the compact groups $P^0(n+1)$ and $P^0(t+1)$ represented in block form are as follows:

$$\begin{pmatrix} J^0 & \mathfrak{A} \\ \mathfrak{P}_{\mathfrak{A}}^{t+1} & J^0 \end{pmatrix}$$

and

$$\begin{pmatrix} J^0 & \mathfrak{A} \\ \mathfrak{P}_{\mathfrak{A}}^{(n+1)} & J^0 \end{pmatrix}$$

respectively. We also define an auxiliary subgroup J'_s:

$$\begin{pmatrix} J^0 & \mathcal{O}_E + \mathfrak{P}_{\mathfrak{A}}^{n-t} \\ \mathfrak{P}_{\mathfrak{A}}^{t+1} & J^0 \end{pmatrix}.$$

Lemma 6.2.1. *The representation* $\mathrm{ind}_{J'_s}^{P^0(t+1)}(\lambda_s)$ *is an irreducible representation of* $P^0(t+1)$.

Proof. The double coset representatives for $J'_s \backslash P^0(t+1)/J'_s$ can be chosen from $U(\mathcal{O}_F) = P^0(t+1) \cap U$. Let u^+ be a coset representative represented in the block diagonal form as

$$u^+ = \begin{pmatrix} \mathrm{id} & U \\ 0 & \mathrm{id} \end{pmatrix}.$$

Suppose T is a non-zero operator in the space

$$\mathrm{Hom}_{J'_s \cap (J'_s)^{u^+}}(\lambda_s, \lambda_s^{u^+}).$$

The operator T satisfies the relation

$$T(\begin{pmatrix} \mathrm{id} & 0 \\ C & \mathrm{id} \end{pmatrix} v) = \begin{pmatrix} \mathrm{id} + UC & -UCU \\ C & -CU + \mathrm{id} \end{pmatrix} T(v).$$

Further we take C in $\mathfrak{P}_{\mathfrak{A}}^{t+1}$. Now we get that

$$\psi_\alpha(\mathrm{id} + UC)\psi_\alpha(-CU + \mathrm{id}) = 1.$$

Hence we have $\psi_{(\alpha U - U\alpha)}(1 + C) = 1$. This shows us that U belongs to $\mathfrak{N}_{-t}(\alpha, \mathfrak{A})$ which is equal to $\mathcal{O}_E + \mathfrak{P}_{\mathfrak{A}}^{n-t}$ (See [BK93, Remark page 42]). Since $J'_s \cap U$ is equal to $\mathcal{O}_E + \mathfrak{P}_{\mathfrak{A}}^{[(n+1)/2]}$, u^+ is equivalent to id. This shows the lemma with Mackey criterion. □

We observe that $P^0(t+1)$ can be decomposed as $(J'_s)P^0(n+1)$ and Mackey decomposition applied to this decomposition shows that the space of intertwining operators

$$\mathrm{Hom}_{P^0(t+1)}(\mathrm{ind}_{P^0(n+1)}^{P^0(t+1)}(\lambda_s), \mathrm{ind}_{J'_s}^{P^0(t+1)}(\lambda_s))$$

has dimension 1. Now we need to find the complement of the representation $\pi_2 := \mathrm{ind}_{J'_s}^{P^0(t+1)}(\lambda_s)$ in $\pi_1 = \mathrm{ind}_{P^0(n+1)}^{P^0(t+1)}(\lambda_s)$.

Let f be an element of the representation π_1. Let $I(f)$ be a function defined by the equation

$$I(f)(p) = \int_{u^- \in \ P^0(t+1) \cap \bar{U}} f(u^- p) du^-$$

for all $p \in P^0(t+1)$.

Lemma 6.2.2. *The operator I is a non-zero intertwining operator between π_1 and π_2.*

Proof. Let $p \in P^0(t+1)$ and $u^+ \in J'_s \cap U$. It is enough to show that $I(f)(u^+p) = \lambda_s(u^+)I(f)(p) = I(f)(p)$. Let u^- and u^+ be represented in 2×2 block matrices as

$$u^- = \begin{pmatrix} 1_2 & 0 \\ U^- & 1_2 \end{pmatrix},\ u^+ = \begin{pmatrix} 1_2 & U^+ \\ 0 & 1_2 \end{pmatrix}$$

respectively. Now observe that

$$\int_{u^- \in\ P^0(t+1)\cap \bar{U}} f(u^-u^+p)du^- = \int_{u^- \in\ P^0(t+1)\cap \bar{U}} f(u^-u^+(u^-)^{-1}u^-p)du^- \tag{6.8}$$

The above integral can be written as

$$\int_{u^- \in\ P^0(t+1)\cap \bar{U}} \psi_{(\alpha U^+ - U^+\alpha)}(1+U^-)f(u^-p)du^-. \tag{6.9}$$

Since $U^+ \in \mathcal{O}_E + \mathfrak{P}_{\mathfrak{A}}^{n-t}$ we get that $\alpha U^+ - U^+\alpha \in \mathfrak{P}_{\mathfrak{A}}^{-t}$ since valuation of α with respect to the filtration $\mathfrak{P}_{\mathfrak{A}}^{k}$, $k \in \mathbb{Z}$, is $-n$. This shows that $I(f) \in \pi_2$. To see that I is non-zero we can take a function $f \in \pi_1$ which is constant on $P^0(t+1) \cap \bar{U}$ and observe that $I(f)(1_4) \neq 0$. $\square$

Notation 6.2. *For an element $u^+ \in U$ the 2×2 block matrix form is always denoted by*

$$\begin{pmatrix} 1_2 & U^+ \\ 0 & 1_2 \end{pmatrix}.$$

Similarly, for any element $u^- \in \bar{U}$ the 2×2 block matrix form is always denoted by

$$\begin{pmatrix} 1_2 & 0 \\ U^- & 1_2 \end{pmatrix}.$$

We note that I is surjective and $\ker(I)$ is the complement of π_1 in π_2. If f is in the kernel of I then the above integral vanishes for all $U^+ \in \mathfrak{A}$. Hence the representation $\ker I$ is contained in the space $S(\alpha)$ given by

$$\left\{ f \in \pi_1 | \int_{u^- \in\ P^0(t+1)\cap \bar{U}} \psi_{(\alpha U^+ - U^+\alpha)}(1+U^-)f(u^-)du^- = 0\ \forall\ \mathrm{U}^+ \in \mathfrak{A} \right\}. \tag{6.10}$$

Lemma 6.2.3. *Let W be an irreducible sub-representation of* $\ker(I)$*. Then irreducible sub-representations of* $\mathrm{ind}_{P^0(t+1)}^{\mathrm{GL}_4(\mathcal{O}_F)}(W)$ *are atypical representations.*

Proof. We first define a subgroup

$$H_t = \begin{pmatrix} 1+\mathfrak{P}_{\mathfrak{A}}^{t+1} & \mathfrak{A} \\ \mathfrak{P}_{\mathfrak{A}}^{t+1} & 1+\mathfrak{P}_{\mathfrak{A}}^{t+1} \end{pmatrix}.$$

of $P^0(t+1)$. Now H_t is a normal subgroup of $P^0(t+1)$ and $P^0(t+1) = H_t P^0(n+1)$. Mackey decomposition gives us

$$\begin{aligned} &\operatorname{res}_{H_t} \operatorname{ind}_{P^0(n+1)}^{P^0(t+1)}(\lambda_s) \\ &\simeq \operatorname{ind}_{P^0(n+1)\cap H_t}^{H_t}(\operatorname{res}_{P^0(n+1)\cap H_t} \lambda_s) \\ &\simeq \operatorname{ind}_{P^0(n+1)\cap H_t}^{H_t}((\psi_\alpha \boxtimes \psi_\alpha)^{\dim \lambda_s}). \end{aligned}$$

The character ψ_α is defined in [BK93][1.1.6]. We first describe the irreducible sub-representations of $\operatorname{ind}_{H_t\cap P^0(n+1)}^{H_t}(\psi_\alpha \boxtimes \psi_\alpha)$.

The group $H_t \cap M$ acts by the character $\psi_\alpha \boxtimes \psi_\alpha$ on the representation $\operatorname{ind}_{H_t\cap P^0(n+1)}^{H_t}(\psi_\alpha \boxtimes \psi_\alpha)$. Note that $(H_t \cap \bar{U})P^0(n+1) = P^0(t+1)$ and by Mackey decomposition we have

$$\operatorname{res}_{H_t\cap\bar{U}} \operatorname{ind}_{H_t\cap P^0(n+1)}^{H_t}(\psi_\alpha \boxtimes \psi_\alpha) = \operatorname{ind}_{P^0(n+1)\cap\bar{U}}^{H_t\cap\bar{U}}(\mathrm{id}).$$

Hence the restriction splits as distinct characters of $(H_t\cap\bar{U})/(P^0(n+1)\cap\bar{U})$. The map $u^- \mapsto U^-$ gives us the isomorphism

$$(H_t\cap\bar{U})/(P^0(n+1)\cap\bar{U}) \simeq \mathfrak{P}_{\mathfrak{A}}^{t+1}/\mathfrak{P}_{\mathfrak{A}}^{n+1}.$$

The group of characters of $\mathfrak{P}_{\mathfrak{A}}^{t+1}/\mathfrak{P}_{\mathfrak{A}}^{n+1}$ is identified in the standard (as in [BK93][1.1.6]) way with the group $\mathfrak{P}_{\mathfrak{A}}^{-n}/\mathfrak{P}_{\mathfrak{A}}^{-t}$.

We can choose a basis $\{f_V | V \in \mathfrak{P}_{\mathfrak{A}}^{-n}/\mathfrak{P}_{\mathfrak{A}}^{-t}\}$ for the space of functions $\operatorname{ind}_{P^0(n+1)\cap\bar{U}}^{H_t\cap\bar{U}}(\mathrm{id})$ such that

$$\begin{pmatrix} \mathrm{id} & 0 \\ U_1 & \mathrm{id} \end{pmatrix} f_V = \psi_V(U_1) f_V$$

where ψ_{U^-} is the character of $\mathfrak{P}_{\mathfrak{A}}^{t+1}/\mathfrak{P}_{\mathfrak{A}}^{n+1}$ corresponding to $U^- \in \mathfrak{P}_{\mathfrak{A}}^{-n}/\mathfrak{P}_{\mathfrak{A}}^{-t}$. Let e_X be the characteristic function for the coset $X + \mathfrak{P}_{\mathfrak{A}}^{n+1}$. The function f_V can be written as

$$f_V = \sum_{X\in\mathfrak{P}_{\mathfrak{A}}^{t+1}/\mathfrak{P}_{\mathfrak{A}}^{n+1}} \psi_V(X) e_X.$$

Let $u^+ \in H_t \cap U$. We first observe that

$$u^+ e_X = \psi_\alpha(1+U^+X)\psi_\alpha(1-XU^+)e_X = \psi_{[\alpha,U^+]}(1+X)e_X.$$

Now

$$\begin{aligned}
&u^+ f_V \\
&= \sum_{X \in \mathfrak{P}_{\mathfrak{A}}^{t+1}/\mathfrak{P}_{\mathfrak{A}}^{n+1}} \psi_V(X) u^+ e_X \\
&= \sum_{X \in \mathfrak{P}_{\mathfrak{A}}^{t+1}/\mathfrak{P}_{\mathfrak{A}}^{n+1}} \psi_V(X) \psi_{[\alpha, U^+]}(1+X) e_X = f_{V+[\alpha,U^+]}.
\end{aligned}$$

Any irreducible sub-representation W of $\mathrm{ind}_{H_t \cap P^0(n+1)}^{H_t}(\psi_\alpha \boxtimes \psi_\alpha)$ contains a character ψ_V for some $V \in \mathfrak{P}_{\mathfrak{A}}^{-n}/\mathfrak{P}_{\mathfrak{A}}^{-t}$. Now consider the space spanned by the set

$$\{f_V, f_{V+[\alpha,U^+]} |\ \forall\ U^+ \in \mathfrak{A}\}.$$

By the observation on the action of the element u^+, we get that the span is stable under the action of $H_t \cap U$ and by construction it is stable under the action of $H_t \cap \bar{U}$. Moreover the group $H_t \cap M$ acts by a character $\psi_\alpha \boxtimes \psi_\alpha$. Hence $W \ = < f_V, f_{V+[\alpha,U^+]} | \forall\ U^+ \in \mathfrak{A} >$. Let us denote the representation W by $W(V)$ where V is a coset representative for $\mathfrak{P}_{\mathfrak{A}}^{-n}/\,\mathrm{img}([\alpha,\ ])$ This description will be sufficient for our present purpose.

Now we return to the proof of the lemma 6.2.3. The subgroup H_t is a normal subgroup of $P^0(t+1)$. Using Clifford theory we can write $\ker(I)$ as a direct sum of irreducible sub-representations:

$$\Gamma := \mathrm{ind}_{Z_{P^0(t+1)}(W(V))}^{P^0(t+1)}\{\widetilde{W(V)}\}$$

for some $V \in \mathfrak{P}_{\mathfrak{A}}^{-n}$ and $\widetilde{W(V)}$ is an irreducible representation of $Z_{P^0(t+1)}(W(V))$. (The representation $\widetilde{W(V)}$ is the isotopic component of $W(V)$ in the representation $\mathrm{ind}_{H_t \cap P^0(n+1)}^{H_t}(\psi_\alpha \boxtimes \psi_\alpha)$. But we do not make use of this.). Let $s_{E/F}$ be a tame co-restriction of A with respect to E (see [BK93][Definition 1.3.3]). Suppose $s_{E/F}(V)$ belongs to $\mathfrak{P}_E^{-t}$ for such a V then $V = [\alpha, U]$ (since the kernel of the map induced by $s_{E/F}$ on $\mathfrak{P}_{\mathfrak{A}}^{-n}/\mathfrak{P}_{\mathfrak{A}}^{-t}$ is given by the image of $[\alpha, \cdot]$ see [BK93][corollary1.4.10]) and hence a contradiction to the identity (6.10). This shows $s_{E/F}(V)$ does not belong to $\mathfrak{P}_E^{-t}$ for all V such that $W(V)$ is contained in Γ.

Note that $(Z_{P^(t+1))}(W(V)))(P^0(t+1) \cap P)$ is equal to $P^0(t+1)$. Mackey decomposition shows that

$$\mathrm{res}_{P^0(t+1)\cap P}\ \mathrm{ind}_{Z_{P^0(t+1)}(W(V))}^{P^0(t+1)}\{\widetilde{W(V)}\} \simeq \mathrm{ind}_{Z_{P^0(t+1)}(W(V))\cap P}^{P^0(t+1)\cap P}\ \widetilde{W(V)}.$$

It follows from Frobenius reciprocity that $\mathrm{Hom}_{P^0(n+1)}(\Gamma, \lambda_s) \neq 0$. In particular $\mathrm{Hom}_{P^0(n+1)\cap P}(\Gamma, \lambda_s) \neq 0$. We note that $P^0(n+1) \cap P$ is equal to

$P^0(t+1) \cap P$ and as a consequence $\mathrm{Hom}_{P^0(t+1) \cap P}(\Gamma, \lambda_s) \neq 0$. Again applying Frobenius reciprocity we note that

$$\mathrm{Hom}_{Z_{P^0(t+1)}(W(V)) \cap P}(\widetilde{W(V)}, \lambda_s) \neq 0.$$

We will construct in the next paragraph another typical representation $\lambda_{s'}$ such that $s' \neq s$ and

$$\mathrm{res}_{Z_{P^0(t+1)}(W(V)) \cap P}(\lambda_s) \simeq \mathrm{res}_{Z_{P^0(t+1)}(W(V)) \cap P}(\lambda_{s'}). \tag{6.11}$$

The group $P^0(n+1) \cap \bar{U}$ acts trivially on Γ and on $\lambda_{s'}$. The condition (6.11) now shows that

$$\mathrm{Hom}_{P^0(n+1)}(\Gamma, \lambda_{s'})$$

is non-zero. Which shows that the representation Γ occurs in the representation

$$\mathrm{ind}_{P^0(t+1)}^{P^0(n+1)}(\lambda_{s'}).$$

Hence the representation $\mathrm{ind}_{P^0(t+1)}^{\mathrm{GL}_4(\mathcal{O}_F)}(\Gamma)$ is atypical for the component s.

Now we give the construction of $\lambda_{s'}$ satisfying equation (6.11). We first bound the group $Z_{P^0(t+1)}(W(V)) \cap M$. The elements of $Z_{P^0(t+1)}(W(V)) \cap M$ act on the characters ψ_V such that

$$\begin{pmatrix} j_1 & 0 \\ 0 & j_2 \end{pmatrix} \psi_V = \psi_{j_1 V j_2^{-1}} = \psi_{V + [\alpha, U^+]}$$

for some $U^+ \in \mathfrak{A}$. We now get that

$$j_1 V j_2^{-1} + \mathfrak{P}_{\mathfrak{A}}^{-t} = V + [\alpha, U^+] + \mathfrak{P}_{\mathfrak{A}}^{-t}. \tag{6.12}$$

Recall that $V \in \mathfrak{P}_{\mathfrak{A}}$, valuation of α with respect to the filtration $\mathfrak{P}_{\mathfrak{A}}^k$, $k \in \mathbb{Z}$, is $-n$ and $U^+ \in \mathfrak{A}$. Applying a tame co-restriction $s_{E/F}$ on both sides of (6.12) and taking j_1 and j_2 in $\mathcal{O}_E^\times$ we get that

$$j_1 s_{E/F}(V) j_2^{-1} + \mathfrak{P}_E^{-t} = s_{E/F}(V) + \mathfrak{P}_E^{-t}.$$

The above equation implies that $j_1 \equiv j_2$ modulo $\mathfrak{P}_E$.

The representation λ_s is $\lambda_1 \boxtimes \lambda_2$ where (J^0, λ_i) are typical representations of σ_i for $i \in \{1, 2\}$. Now J^0/J^1 is isomorphic to $k_E^\times$. Since $\# k_E > 3$ we can choose a nontrivial character η of $k_E^\times$ such that the multi-set of types $\{\lambda_1, \lambda_2\}$ and $\{\lambda_1\eta, \lambda_2\eta^{-1}\}$ are distinct. Let $\lambda_{s'} = \lambda_1\eta \boxtimes \lambda_2\eta^{-1}$. The pair (J_s, λ'_s) is a type for the component $s' = [\mathrm{GL}_2(F) \times \mathrm{GL}_2(F), \sigma'_1 \boxtimes \sigma'_2]$ where σ'_1 and σ'_2 are supercuspidal representations containing $(J^0, \lambda\eta)$ and $(J^0, \lambda\eta^{-1})$ respectively. Moreover we observe that $s' \neq s$ and

$$\mathrm{res}_{Z_{P^0(t+1)}(W(V)) \cap P}(\lambda_s) \simeq \mathrm{res}_{Z_{P^0(t+1)}(W(V)) \cap P}(\lambda_{s'}).$$

$\square$

From the above lemma typical representations for the component s occur as sub-representations of

$$\mathrm{ind}_{J'_s}^{\mathrm{GL}_4(\mathcal{O}_F)}(\lambda_s).$$

Since J_s contains the group J'_s we have

$$\mathrm{ind}_{J_s}^{\mathrm{GL}_4(\mathcal{O}_F)}\{\mathrm{ind}_{J'_s}^{J_s}(\mathrm{id})\otimes\lambda_s\}\simeq\mathrm{ind}_{J'_s}^{\mathrm{GL}_4(\mathcal{O}_F)}(\lambda_s).$$

Frobenius reciprocity implies that $\mathrm{ind}_{J'_s}^{J_s}(\mathrm{id})$ contains id with multiplicity one. Let $\epsilon(s)$ be the complement of id in $\mathrm{ind}_{J'_s}^{J_s}(\mathrm{id})$.

Lemma 6.2.4. *Irreducible sub-representations of*

$$\mathrm{ind}_{J_s}^{\mathrm{GL}_4(\mathcal{O}_F)}\{\epsilon(s)\otimes\lambda_s\}$$

are not typical representations.

Proof. The subset $\mathfrak{P}_E\cap\mathfrak{P}_{\mathfrak{A}}^{t+1}$ is an ideal in $\mathcal{O}_E$ and $\mathfrak{P}_E\cap\mathfrak{P}_{\mathfrak{A}}^{t+1}=\mathfrak{P}_E^{t+1}$. We first define a sequence of subgroups of J_s as follows:

$$H_i=\begin{pmatrix}J^0 & \mathcal{O}_E+\mathfrak{P}_{\mathfrak{A}}^{n-t}\\ \mathfrak{P}_E^i+\mathfrak{P}_{\mathfrak{A}}^{t+1} & J^0\end{pmatrix}\ \forall\ 1\leq i\leq t+1.$$

We note that that $H_1=J_s$ and $H_{t+1}=J'_s$. Now

$$\mathrm{ind}_{H_{(i+1)}}^{\mathrm{GL}_4(\mathcal{O}_F)}(\lambda_s)=\mathrm{ind}_{H_i}^{\mathrm{GL}_4(\mathcal{O}_F)}(\lambda_s)\oplus\mathrm{ind}_{H_i}^{\mathrm{GL}_4(\mathcal{O}_F)}(\epsilon_i(s)\otimes\lambda_s).$$

Where $\epsilon_i(s)$ is the complement of id in the representation $\mathrm{ind}_{H_{(i+1)}}^{H_i}(\mathrm{id})$. If $\mathrm{ind}_{H_i}^{\mathrm{GL}_4(\mathcal{O}_F)}(\lambda_s)$ admits a complement of $\mathrm{ind}_{H_1}^{\mathrm{GL}_4(\mathcal{O}_F)}(\lambda_s)$ containing only atypical representations then we show the same for $i+1$. For this we verify that irreducible sub-representations of $\mathrm{ind}_{H_i}^{\mathrm{GL}_4(\mathcal{O}_F)}(\epsilon_i(s)\otimes\lambda_s)$ are atypical for the component s. Hence by induction we show the above lemma. Consider the representation

$$\mathrm{ind}_{H_{(i+1)}}^{H_i}(\mathrm{id}).$$

The group $H_i\cap U$ acts trivially on this representation. To see this let u^+ be an element of $H_i\cap U$. We can choose coset representatives u^- for $H_i/H_{(i+1)}$ from $H_i\cap\bar{U}$. Let u^+ and u^- represented in their block form be

$$\begin{pmatrix}1_2 & U^+\\ 0 & 1_2\end{pmatrix}\ \begin{pmatrix}1_2 & 0\\ U^- & 1_2\end{pmatrix}$$

respectively. Moreover we may assume that $U^-\in\mathfrak{P}_E^i$. We observe that $(u^-)^{-1}u^+u^-$ is of the form

$$\begin{pmatrix}1_2-U^+U^- & U^+\\ -U^-U^+U^- & 1_2+U^-U^+\end{pmatrix}$$

and this is clearly in the group $H_{(i+1)}$. Hence u^+ acts trivially on $\mathrm{ind}_{H_{(i+1)}}^{H_i}(\mathrm{id})$.

Define $S(i) = H_i \cap \bar{P}$ where $\bar{P}$ is the opposite group of P with respect to the Levi-subgroup M. Observe that

$$\mathrm{res}_{H_i \cap \bar{U}} \mathrm{ind}_{H_{(i+1)}}^{H_i}(\mathrm{id}) = \oplus \eta_i$$

where η_i are distinct characters of $(H_i \cap \bar{U})/(H_{(i+1)} \cap \bar{U})$. Now the group $S(i) \cap M$ acts on these characters and splits them into two different orbits id and the rest of the characters. Applying Clifford theory we have

$$\mathrm{res}_{S(i)} \mathrm{ind}_{H_{(i+1)}}^{H_i}(\mathrm{id}) = \mathrm{id} \oplus \mathrm{ind}_{Z_{S(i)}(\eta)}^{S(i)}(U_\eta)$$

where η is a nontrivial character of $(H_i \cap \bar{U})/(H_{(i+1)} \cap \bar{U})$ and U_η is an irreducible representation of the group $Z_{S(i)}(\eta)$. This decomposition extends to a representation of H_i.

For all $U^- \in \mathfrak{P}_E^i$, the element

$$u^- = \begin{pmatrix} 1_2 & 0 \\ U^- & 1_2 \end{pmatrix}$$

is contained in the group $H_i \cap \bar{U}$. Now the map $U^- \mapsto u^-$ gives us an isomorphism

$$\mathfrak{P}_E^i / \mathfrak{P}_E^{i+1} \simeq (H_i \cap \bar{U})/(H_{i+1} \cap \bar{U}).$$

The above isomorphism is $\mathcal{O}_E^\times \times \mathcal{O}_E^\times$ (considered as a subgroup of M) equivariant.

Let

$$\begin{pmatrix} j_1 & 0 \\ 0 & j_2 \end{pmatrix}$$

be an element in $Z_{S(i)}(\eta) \cap M$ and j_1, j_2 belong to $\mathcal{O}_E^\times$. Since η is non-trivial we have $j_1 \equiv j_2$ modulo $\mathfrak{P}_E$. As in the previous lemma we can construct another component s' such that $s' \neq s$ and

$$\mathrm{res}_{Z_{S(i)}(\eta)}(\lambda_s) = \mathrm{res}_{Z_{S(i)}(\eta)}(\lambda_{s'}).$$

This shows us that the irreducible sub-representations of

$$\mathrm{ind}_{H_i}^{\mathrm{GL}_2(\mathcal{O}_F)}(\mathrm{ind}_{Z_{S(i)}(\eta)}^{S(i)}(U_\eta) \otimes \lambda_s) \simeq \mathrm{ind}_{H_i}^{\mathrm{GL}_2(\mathcal{O}_F)}(\epsilon_i(s) \otimes \lambda_s)$$

are not typical representations. Hence we show the lemma. $\square$

With this discussion we conclude that all typical representations for the component s occur as sub-representations of

$$\mathrm{ind}_{J_s}^{\mathrm{GL}_4(\mathcal{O}_F)}(\lambda_s).$$

6.2.2 The complete classification when E is ramified

Although ideas in this section are essentially inspired from previous section, we have to do additional work at various instances to finish the complete classification of typical representations when $e(\mathfrak{A}) = 2$.

Let $\mathfrak{M}$ and $\mathfrak{J}$ be two hereditary orders corresponding to the lattice chains $\mathcal{L}_1(i) = {\mathfrak{P}_F}^i \oplus {\mathfrak{P}_F}^i$ and $\mathcal{L}_2(2i) = {\mathfrak{P}_F}^i \oplus {\mathfrak{P}_F}^i, \mathcal{L}_2(2i+1) = {\mathfrak{P}_F}^i \oplus {\mathfrak{P}_F}^{i+1}$ respectively. Let $[\mathfrak{J}, 2n-1, 0, \alpha]$ be a simple strata contained in σ_1 and σ_2. We have $v_{\mathfrak{J}}(\alpha) = -(2n-1)$ (the valuation given by the filtration $\mathfrak{P}_{\mathfrak{J}}^k$, $k \in \mathbb{Z}$) and from the inclusions

$$\mathfrak{P}_{\mathfrak{M}}^{-(n-1)} \subset \mathfrak{P}_{\mathfrak{J}}^{-(2n-2)} \subset \mathfrak{P}_{\mathfrak{J}}^{-(2n-1)} \subset \mathfrak{P}_{\mathfrak{A}}^{-n}$$

we get that $v_{\mathfrak{M}}(\alpha) = -n$. Let $s_{E/F}$ be a tame co-restriction map. Now

$$s_{E/F}(\mathfrak{P}_{\mathfrak{M}}^i) = s_{E/F}(\varpi_{\mathfrak{J}}^{2i}\mathfrak{M}) = \varpi_E^{2i+r}\mathcal{O}_E$$

where r is given by $s_{E/F}(\mathfrak{M}) = \mathfrak{P}_E^r$. Hence we have a sequence

$$\frac{\mathfrak{M}}{\mathcal{O}_E + \mathfrak{P}_{\mathfrak{M}}} \xrightarrow{[\alpha,\ ]} \frac{\mathfrak{P}_{\mathfrak{M}}^{-n}}{\mathfrak{P}_{\mathfrak{M}}^{-(n-1)}} \xrightarrow{s_{E/F}} \frac{\mathfrak{P}_E^{-2n+r}}{\mathfrak{P}_E^{-2(n-1)+r}} \tag{6.13}$$

Lemma 6.2.5. *The sequence (6.13) is exact.*

Proof. The composition of the maps

$$\frac{\mathfrak{M}}{\mathfrak{P}_{\mathfrak{M}}} \xrightarrow{[\alpha,\ ]} \frac{\mathfrak{P}_{\mathfrak{M}}^{-n}}{\mathfrak{P}_{\mathfrak{M}}^{-(n-1)}} \xrightarrow{\varpi_F^n} \frac{\mathfrak{M}}{\mathfrak{P}_{\mathfrak{M}}}$$

is given by a map $m \mapsto [\overline{\varpi_F^n\alpha}, m]$ from $M_2(k_F)$ to $M_2(k_F)$. Here $\overline{\varpi^n\alpha}$ is the class of $\varpi_F^n\alpha$ in $\mathfrak{M}/\mathfrak{P}_{\mathfrak{M}}$. We have $\varpi_F^{2n}\alpha^2 = \varpi_{\mathfrak{J}}^{4n}\alpha^2 = \varpi_F(\varpi_{\mathfrak{J}}^{4n-2}\alpha^2)$ and $\varpi_{\mathfrak{J}}^{4n-2}\alpha^2 \in \mathfrak{J}$. This shows that the $\varpi_F^n\alpha$ is a nontrivial nilpotent matrix. The dimension of the commutator of a nontrivial nilpotent element is 2. Now $\mathcal{O}_E$ is in the kernel and we observe that $\mathcal{O}_E/\varpi_F\mathcal{O}_E$ is a 2 dimensional vector space over k_F. This shows that kernel is exactly $\mathcal{O}_E + \mathfrak{P}_{\mathfrak{M}}$. Since the image of $[\alpha,\ ]$ in the sequence (6.13) is of dimension 2 we get that $\mathrm{img}[\alpha,\ ] = \ker s_{E/F}$. This concludes the proof of the lemma. ☐

The proof of the above lemma shows that $\mathfrak{N} = \{u \in \mathfrak{M} \mid [\alpha, u] \in \mathfrak{P}_{\mathfrak{M}}^{-(n-1)}\} = \mathcal{O}_E + \mathfrak{P}_{\mathfrak{M}}$. We denote by H_1, H_2 the groups

$$\begin{pmatrix} J^0 & \mathfrak{M} \\ \mathfrak{P}_{\mathfrak{M}}^{n+1} & J^0 \end{pmatrix} \text{ and } \begin{pmatrix} J^0 & \mathfrak{M} \\ \mathfrak{P}_{\mathfrak{M}}^{n} & J^0 \end{pmatrix}.$$

Note that these groups are $P^0(n+1)$ and $P^0(n)$ defined in the previous section. We note that the group $J^0 \times J^0$ (considered as a subgroup of $M(\mathcal{O}_F)$) normalizes the group

$$U(\mathcal{O}_E + \mathfrak{P}_{\mathfrak{M}}) = \left\{ \begin{pmatrix} 1_2 & B \\ 0 & 1_2 \end{pmatrix} \mid B \in \mathcal{O}_E + \mathfrak{P}_{\mathfrak{M}} \right\}$$

hence we have the semi-direct product $(J^0 \times J^0)U(\mathcal{O}_E + \mathfrak{P}_{\mathfrak{M}})$. Now we define H_3 to be the subgroup $K_4(n)(J^0 \times J^0)U(\mathcal{O}_E + \mathfrak{P}_{\mathfrak{M}})$. The group H_3 in the block form is as follows :

$$\begin{pmatrix} J^0 & \mathcal{O}_E + \mathfrak{P}_{\mathfrak{M}} \\ \mathfrak{P}_{\mathfrak{M}}^n & J^0 \end{pmatrix}.$$

Lemma 6.2.6. *The representation λ_s of $H_3 \cap M$ extends to a representation of H_3 such that $H_3 \cap U$ and $H_3 \cap \bar{U}$ are contained in the kernel of the extended representation.*

Proof. The representation λ_s extends to a representation of

$$H_3' = \begin{pmatrix} J^0 & \mathcal{O}_E + \mathfrak{P}_{\mathfrak{M}} \\ \mathfrak{P}_{\mathfrak{M}}^{n+1} & J^0 \end{pmatrix}$$

since $\mathrm{res}_{1_2 + \mathfrak{P}_{\mathfrak{M}}^{n+1}} \lambda_s = \mathrm{id}$. Now $H_3 = H_3'(H_3 \cap \bar{U})$. Let

$$u^+ = \begin{pmatrix} 1_2 & B \\ 0 & 1_2 \end{pmatrix}$$

be an element of $(H_3' \cap U) = (H_3 \cap U)$ and

$$u^- = \begin{pmatrix} 1_2 & 0 \\ C & 1_2 \end{pmatrix}$$

be an element of $H_3 \cap \bar{U}$. We observe that $u^- u^+ {u^-}^{-1}$ is of the form

$$\begin{pmatrix} 1_2 - BC & B \\ -CBC & CB + 1_2 \end{pmatrix}.$$

The above element belongs to the group H_3' and

$$\lambda_s(u^- u^+ {u^-}^{-1}) = \psi_{[\alpha,B]}(1 + C) = 1.$$

Hence the representation λ_s extends to a representation of H_3 such that $H_3 \cap \bar{U}$ is contained in the kernel of the extension. $\square$

By Mackey decomposition we get that

$$\dim_{\mathbb{C}} \mathrm{Hom}_{H_2}(\mathrm{ind}_{H_1}^{H_2}(\lambda_s), \mathrm{ind}_{H_3}^{H_2}(\lambda_s)) = 1.$$

Let π_1 and π_2 be the representations $\mathrm{ind}_{H_1}^{H_2}(\lambda_s)$ and $\mathrm{ind}_{H_3}^{H_2}(\lambda_s)$ respectively. Let I be an operator from the space π_1 to the space of functions on H_1 given by

$$I(f)(h) = \int_{H_1 \cap \bar{U}} f(uh)du.$$

Lemma 6.2.7. *The operator I is a non-trivial intertwining operator from π_1 to π_2.*

Lemma 6.2.8. *The representation $\mathrm{ind}_{H_3}^{H_2}(\lambda_s)$ is an irreducible representation of H_2.*

Lemma 6.2.9. *The irreducible sub-representations of $\mathrm{ind}_{H_2}^{\mathrm{GL}_4(\mathcal{O}_F)}(\ker(I))$ are not typical representations.*

We prove lemmas 6.2.7, 6.2.8 and 6.2.9 in an axiomatic way as the same argument is used in various contexts. Let us first recall some definitions (see [BK93][1.1.4]). Let ψ be an additive character of F which is trivial on $\mathfrak{P}_F$ but not on $\mathcal{O}_F$ and A be the set of matrices $M_2(F)$. We denote by ψ_A the character $x \mapsto \psi(\mathrm{tr}(x))$. For a given subset $S \subset A$, we denote by S^* the set

$$\{x \in A \mid \psi_A(xs) = 0 \ \forall\ s \in S\}.$$

Let $(H_1, H_2, H_3, \lambda_s)$ be a tuple consisting of three groups H_i for $1 \leq i \leq 3$ and λ_s a representation of a common subgroup of H_i for $1 \leq i \leq 3$. We assume that the tuple satisfies the following conditions

1. $H_1 = \begin{pmatrix} J^0 & \mathfrak{N}_1 \\ \mathfrak{P}_2 & J^0 \end{pmatrix}$.

2. $H_2 = \begin{pmatrix} J^0 & \mathfrak{N}_1 \\ \mathfrak{P}_1 & J^0 \end{pmatrix}$.

3. $H_3 = \begin{pmatrix} J^0 & \mathfrak{N}_2 \\ \mathfrak{P}_1 & J^0 \end{pmatrix}$.

4. J^0 is the group J^0_α for the simple strata $[\mathfrak{A}, n, 0, \alpha]$ and λ_s is a representation $\lambda_1 \boxtimes \lambda_2$ of $J^0 \times J^0$ such that (J^0, λ_1) and (J^0, λ_2) are Bushnell-Kutzko types for some supercuspidal representations σ_1 and σ_2 respectively.

5. The lattices $\mathfrak{P}_i$ and $\mathfrak{N}_i$ for i in $\{1, 2\}$ satisfy the inclusion relations $\mathfrak{P}_2 \subseteq \mathfrak{P}_1$, $\mathfrak{P}_1\mathfrak{P}_1 \subseteq \mathfrak{P}_2$, $\mathfrak{N}_2 \subseteq \mathfrak{N}_1$ and $\mathfrak{P}_1\mathfrak{N}_1\mathfrak{P}_1 \subseteq \mathfrak{P}_2$. The map ψ_α sending x to $\mathrm{tr}(\alpha x)$ is a character of the quotient $\mathfrak{P}_1/\mathfrak{P}_2$ where α is in $\mathfrak{P}_2^*$. λ_s extends to a representation of H_1 and H_3.

6. λ_1 and λ_2 are representations of J^0 such that the restriction to $1_2 + \mathfrak{P}_1$ is a multiple of the character of the form $x \mapsto \psi_\alpha(1 + x)$

7. We have an exact sequence

$$0 \to \frac{\mathfrak{N}_1}{\mathfrak{N}_2} \to \frac{\mathfrak{P}_2^*}{\mathfrak{P}_1^*} \to \frac{s(\mathfrak{P}_2^*)}{s(\mathfrak{P}_1^*)} \to 0 \tag{6.14}$$

such that the first arrow is given by $[\alpha, \ ]$ and the second by s the tame co-restriction with respect to the field $F[\alpha]/F$.

It follows from Mackey decomposition that up to scalars there exists a unique non-trivial intertwining operator between the representations $\pi_1 = \mathrm{ind}_{H_1}^{H_2}(\lambda_s)$ and $\pi_2 = \mathrm{ind}_{H_3}^{H_2}(\lambda_s)$. Let f be a function in the space π_1. Let I be the operator from π_1 to the space of functions on H_1 given by

$$I(f)(h) = \int_{H_2 \cap \bar{U}} f(uh)du$$

for all $h \in H_1$.

Lemma 6.2.10. *The operator I is a non-trivial intertwining operator between the space π_1 and π_2.*

Proof. The proof is a repetition of the proof of 6.2.2. Let $h \in H_2$ and $u^+ \in H_3 \cap U$. It is enough to show that $I(f)(u^+h) = \lambda_s(u^+)I(f)(h) = I(f)(h)$. Let u^- and u^+ be represented in 2×2 block matrices as

$$u^- = \begin{pmatrix} 1_2 & 0 \\ U^- & 1_2 \end{pmatrix}, \ u^+ = \begin{pmatrix} 1_2 & U^+ \\ 0 & 1_2 \end{pmatrix}$$

respectively.

$$\int_{u^- \in \ H_1 \cap \bar{U}} f(u^-u^+p)du^- = \int_{u^- \in \ H_1 \cap \bar{U}} f(u^-u^+(u^-)^{-1}u^-p)du^- \tag{6.15}$$

Using the axiom 6 the above integral can be written as

$$\int_{u^- \in \ H_1 \cap \bar{U}} \psi_{(\alpha U^+ - U^+ \alpha)}(1 + U^-)f(u^-p)du^-. \tag{6.16}$$

Since $U^+ \in \mathfrak{N}_2$ and by the exact sequence in axiom (7) we get that $\alpha U^+ - U^+\alpha$ belongs to $\mathfrak{P}_1^*$. This shows that $I(f) \in \pi_2$. To see that I is a non-zero operator we can take a function $f \in \pi_1$ which is a non-zero constant on $H_1 \cap \bar{U}$ and observe that $I(f)(1_4) \neq 0$. □

Lemma 6.2.11. *The representation π_2 is an irreducible representation of H_2. Any irreducible sub-representation in the kernel of this intertwining operator I also occurs as a sub-representation of*

$$\mathrm{ind}_{H_1}^{H_2}(\lambda_1\eta \boxtimes \lambda_2\eta^{-1})$$

for any tame character η of $\mathcal{O}_E^\times$.

Proof. Except for few changes the proof is a repetition of lemma 6.2.3. We first prove that π_2 is irreducible. The double coset representatives for $H_3\backslash H_2/H_3$ can be chosen from $H_2 \cap U$. Let u^+ be a coset representative. We write u^+ in block diagonal form as

$$u^+ = \begin{pmatrix} \mathrm{id} & U \\ 0 & \mathrm{id} \end{pmatrix}.$$

Suppose T be a non-zero operator in the space

$$\mathrm{Hom}_{H_3 \cap (H_3)^{u^+}}(\lambda_s, \lambda_s^{u^+}).$$

The operator T satisfies the relation

$$T(\begin{pmatrix} \mathrm{id} & 0 \\ C & \mathrm{id} \end{pmatrix} v) = \begin{pmatrix} \mathrm{id} + UC & -UCU \\ C & -CU + \mathrm{id} \end{pmatrix} T(v).$$

where $C \in \mathfrak{P}_1$. Now we get that

$$\psi_\alpha(\mathrm{id} + UC)\psi_\alpha(-CU + \mathrm{id}) = 1.$$

Hence we have $\psi_{(\alpha U - U\alpha)}(1 + C) = 1$ for all $C \in \mathfrak{P}_1$. The first arrow of the exact sequence (6.14) shows that $U \in \mathfrak{N}_2$ and hence u^+ belongs the double coset represented by id. By Mackey irreducibility criteria π_2 is irreducible. For the second part let f be a function in $\ker(I)$ then we have

$$I(f)(u^+) = \int_{u^- \in H_2 \cap \bar{U}} f(u^- u^+) du^- = 0$$

for all u^+ in $H_2 \cap U$. We write u^+ and u^- in their block diagonal form as

$$\begin{pmatrix} \mathrm{id} & U^+ \\ 0 & \mathrm{id} \end{pmatrix}$$

and

$$\begin{pmatrix} \mathrm{id} & U^- \\ 0 & \mathrm{id} \end{pmatrix}$$

respectively. We observe that $u^- u^+ {u^-}^{-1}$ is of the form

$$\begin{pmatrix} 1_2 - U^+U^- & U^+ \\ -U^-U^+U^- & U^-U^+ + 1_2 \end{pmatrix}.$$

The above matrix is an element of H_1. We now have

$$\begin{aligned} &\int_{H_2 \cap \bar{U}} f(u^- u^+) du^- \\ &= \int_{H_2 \cap \bar{U}} \lambda_s(u^- u^+ {u^-}^{-1}) f(u^-) du^- \\ &= \int_{U^- \in \mathfrak{P}_1} \psi_\alpha(1_2 - U^+U^-)\psi_\alpha(1_2 + U^-U^+) f(U^-) dU^-. \end{aligned}$$

This shows that $\ker(I)$ consists of functions which satisfy the identity

$$\int_{U^-\in\mathfrak{P}_1} \psi_{(\alpha U^+ - U^+\alpha)}(1+U^-)f(U^-)dU^- = 0$$

for all $U^+ \in \mathfrak{N}_1$. We denote by H' the group

$$\begin{pmatrix} 1_2+\mathfrak{P}_1 & \mathfrak{N}_1 \\ \mathfrak{P}_1 & 1_2+\mathfrak{P}_1 \end{pmatrix}.$$

We have $H'H_1 = H_2$ hence by Mackey decomposition we get that

$$\operatorname{res}_{H'} \operatorname{ind}_{H_1}^{H_2}(\lambda_s) = \operatorname{ind}_{H_1\cap H'}^{H'}(\lambda_s).$$

Let $\operatorname{res}_{H'\cap M} \lambda_s$ be $(\psi_\alpha \boxtimes \psi_\alpha)^n$. Now we get that

$$\operatorname{ind}_{H_1\cap H'}^{H'}(\lambda_s) \simeq (\operatorname{ind}_{H_1\cap H'}^{H'}(\psi_\alpha \boxtimes \psi_\alpha))^n.$$

The representation

$$\operatorname{ind}_{H_1\cap H'}^{H'}(\psi_\alpha \boxtimes \psi_\alpha)$$

can be realised as space of functions on the abelian group

$$\frac{H'}{H_1\cap H'} \simeq \frac{\mathfrak{P}_1}{\mathfrak{P}_2} \tag{6.17}$$

If e_{u^-} is the characteristic function for the coset representative u^-, the element u^+ acts by the constant $\psi_{\alpha U^+ - U^+\alpha}(U^-)$ on e_{u^-}. The element $j = \operatorname{diag}(j_1, j_2)$ of $H' \cap M$ acts by sending e_{U^-} to

$\psi_\alpha(j_1 j_2) e_{j_1 u^- j_2^{-1}} = \psi_\alpha(j_1 j_2) e_{u^-}$ (see 6.12). The space of functions on the group (6.17) are spanned by the characters on the abelian quotient (6.17). The set of characters of the group (6.17) can be identified with the standard isomorphism

$$\frac{\mathfrak{P}_2^*}{\mathfrak{P}_1^*} \simeq \widehat{\frac{\mathfrak{P}_1}{\mathfrak{P}_2}}$$

sending U to ψ_U. The action of the element u^+ on ψ_V is given by

$$\psi_V \mapsto \psi_{V+[\alpha,U^+]}.$$

Let $W(V)$ be a space spanned by the functions of the form $\psi_{V+[\alpha,U^+]}$ for all $U^+ \in \mathfrak{N}_1$. This space is stable for the action of H' and is irreducible. Hence we have

$$\operatorname{ind}_{H_1\cap H'}^{H'}(\lambda_s) \simeq \bigoplus_{V\in\mathfrak{P}_2^*/\operatorname{img}([\alpha,\ ])} (W(V))^n$$

Now the group H_2 acts on the set of representations $W(V)$ of H' and we get that

$$\operatorname{ind}_{H_1}^{H_2}(\lambda_s) \simeq \bigoplus_{V_i} (\operatorname{ind}_{Z(W(V_i))}^{H_2}(\widetilde{W(V_i)}))^n$$

where $V_i | 1 \leq i \leq l$ is the representatives for the action of H_2, $Z(W(V_i))$ is the stabilizer of the group $W(V_i)$ and $\widetilde{W(V_i)}$ is the isotopic component of $W(V_i)$.

Let $s_{E/F}$ be a tame co-restriction on A with respect to E (see [BK93][1.3.3]) If $s_{E/F}(V_i) = 0$ then by the exact sequence (6.14) we get that $V_i = [\alpha, U^+]$ for some $U^+ \in \mathfrak{N}_1/\mathfrak{N}_2$. This shows that the representation

$$\Gamma := \mathrm{ind}_{Z(W(V_i))}^{H_2}(\widetilde{W(V_i)})$$

contains a function $\psi_{[\alpha,U^+]}$. Hence the above representation is not contained in the space $\ker(I)$. Hence the kernel consists of representations $W(V)$ with $s_{E/F}(V) \neq 0$ Now $Z(W(V)) = \{(J^0 \times J^0) \cap Z(W(V))\}H'$. Note that $J^0 = \mathcal{O}_E^\times J^1$ and $(\mathcal{O}_E^\times \times \mathcal{O}_E^\times) \cap \{(J^0 \times J^0) \cap Z(W(V))\}$ is contained in the group of the form $\{(a,b) \mid a, b \in \mathcal{O}_E^\times; a \equiv b \bmod \mathfrak{P}_E\}$. Let η be a character of $J^0/J^1 \simeq k_E^\times$. We observe that

$$\mathrm{res}_{Z(W(V))} \lambda_1 \boxtimes \lambda_2 \simeq \mathrm{res}_{Z(W(V))} \lambda_1\eta \boxtimes \lambda_2\eta^{-1}$$

Now we note that $(Z(W(V)) \cap P)H_1 = H_2$ and $H_1 \cap P = H_2 \cap P$. From Frobenius reciprocity we get that $\mathrm{Hom}_{H_2 \cap P}(\Gamma, \lambda_s) \neq 0$. We have

$$\begin{aligned}
&\mathrm{Hom}_{H_2 \cap P}(\Gamma, \lambda_{s'}) \\
&= \mathrm{Hom}_{Z(W(V)) \cap P}(\widetilde{W(V)}, \lambda_{s'}) \\
&= \mathrm{Hom}_{Z(W(V)) \cap P}(\widetilde{W(V)}, \lambda_s).
\end{aligned}$$

and $\mathrm{Hom}_{Z(W(V)) \cap P}(\widetilde{W(V)}, \lambda_s)$ is equal to $\mathrm{Hom}_{H_2 \cap P}(\Gamma, \lambda_s)$ which shows that $\mathrm{Hom}_{H_1}(\Gamma, \lambda_{s'})$ is non-zero. □

Proof of lemma 6.2.8 and 6.2.9. We apply lemma 6.2.11 for the tuple

$$(H_1, H_2, H_3, \lambda_s)$$

defined at the beginning of this subsection with (J^0, λ_1) and (J^0, λ_2) being the Bushnell-Kutzko types for the representations σ_1 and σ_2 respectively. The exact sequence is provided by (6.13). The irreducible sub-representations of $\ker(I)$ also occur in the representation

$$\mathrm{ind}_{H_1}^{H_2}(\lambda_1\eta \boxtimes \lambda_2\eta^{-1}).$$

for any tame character η of $\mathcal{O}_E^\times$. If $\# k_F > 3$ then we can choose a character η of $k_E^\times = k_F^\times$ such that the multi-sets of types $\{\lambda_1, \lambda_2\}$ and $\{\lambda_1\eta, \lambda_2\eta^{-1}\}$ are distinct. If σ_1' and σ_2' are two supercuspidal representations containing $(J^0, \lambda_1\eta)$ and $(J^0, \lambda_2\eta^{-1})$ respectively then $s' = [M, \sigma_1' \boxtimes \sigma_2']$ and $s = [M, \sigma_1 \boxtimes \sigma_2]$ are different inertial classes and hence the irreducible sub-representations of $\mathrm{ind}_{H_2}^{\mathrm{GL}_4(\mathcal{O}_F)}(\ker(I))$ occur as sub-representations of $\mathrm{ind}_{H_2}^{\mathrm{GL}_4(\mathcal{O}_F)}(\lambda_1\eta \boxtimes \lambda_2\eta^{-1})$ and hence are atypical. □

Now typical representations occur as sub-representations of $\mathrm{ind}_{H_3}^{\mathrm{GL}_4(\mathcal{O}_F)}(\lambda_s)$ or rather the complement of this representation in the parabolic induction

$$i_P^{\mathrm{GL}_4(F)}(\sigma_1 \boxtimes \sigma_2)$$

contains only atypical representations for the component s.

Lemma 6.2.12. *The group $\mathcal{O}_E + \mathfrak{P}_{\mathfrak{J}}$ is equal to $\mathcal{O}_E + \mathfrak{P}_{\mathfrak{M}}$.*

Proof. Since the group group $\mathfrak{P}_{\mathfrak{M}}$ is contained in $\mathfrak{P}_{\mathfrak{J}}$, $\mathcal{O}_E + \mathfrak{P}_{\mathfrak{J}}$ is a subset of $\mathcal{O}_E + \mathfrak{P}_{\mathfrak{M}}$. We will now show that $\mathfrak{P}_{\mathfrak{J}} \subset \mathcal{O}_E + \mathfrak{P}_{\mathfrak{M}}$. We recall that the groups $\mathfrak{P}_{\mathfrak{J}}$ and $\mathfrak{P}_{\mathfrak{M}}$ are given by

$$\begin{pmatrix} \mathfrak{P}_F & \mathcal{O}_F \\ \mathfrak{P}_F & \mathfrak{P}_F \end{pmatrix} \text{ and } \begin{pmatrix} \mathfrak{P}_F & \mathfrak{P}_F \\ \mathfrak{P}_F & \mathfrak{P}_F \end{pmatrix}$$

respectively. Let Π be the matrix of the form

$$\begin{pmatrix} 0 & 1 \\ \varpi_F & 0 \end{pmatrix}.$$

We note that

$$\mathcal{O}_F \Pi + \mathfrak{P}_{\mathfrak{M}} = \mathfrak{P}_{\mathfrak{J}}. \tag{6.18}$$

Since the element Π normalizes the hereditary order $\mathfrak{J}$, we have $\Pi = \varpi_E j$ for some $j \in U(\mathfrak{J})$. Now multiplying j^{-1} on both sides of the equation (6.18) we get that $\varpi_E \mathcal{O}_F + \mathfrak{P}_{\mathfrak{M}} = \mathfrak{P}_{\mathfrak{J}}$. This shows the lemma. □

Hence the group H_3 is of the following form.

$$\begin{pmatrix} J^0 & \mathcal{O}_E + \mathfrak{P}_{\mathfrak{J}} \\ \mathfrak{P}_{\mathfrak{M}}^n & J^0 \end{pmatrix}.$$

Let a, b be integers such that $a + b = 2n$, $a \geq n$ and $b \geq 0$. We denote by $\mathcal{H}(a, b)$ the set consisting of the matrices

$$\left\{ \begin{pmatrix} A & B \\ C & D \end{pmatrix} A, D \in \mathcal{O}_E + \mathfrak{P}_{\mathfrak{J}}^n;\ B \in \mathcal{O}_E + \mathfrak{P}_{\mathfrak{J}}^b; C \in \mathfrak{P}_{\mathfrak{J}}^a \right\}.$$

Lemma 6.2.13. *The set $\mathcal{H}(a, b)$ is an order.*

Proof. Let h_1 and h_2 be two matrices from the set $\mathcal{H}(a, b)$ we write h_1 and h_2 in its block form as

$$\begin{pmatrix} A_1 & B_1 \\ C_1 & D_1 \end{pmatrix} \quad \begin{pmatrix} A_2 & B_2 \\ C_2 & D_2 \end{pmatrix}$$

The product of $h_1 h_2$ has the 1×1 entry $A_1 A_2 + B_1 C_2$ Since $a + b = 2n$ we can see that $B_1 C_2 \in \mathcal{O}_E + \mathfrak{P}_{\mathfrak{J}}^n + \mathfrak{P}_{\mathfrak{J}}^a$ since we assumed $a \geq n$ we have $B_1 C_2 \in \mathfrak{P}_{\mathfrak{J}}^n$ and hence $A_1 A_2 + B_1 C_2 \in \mathcal{O}_E + \mathfrak{P}_{\mathfrak{J}}^n$. The 1×2 term is contained in $(\mathcal{O}_E + \mathfrak{P}_{\mathfrak{J}}^n)(\mathcal{O}_E + \mathfrak{P}_{\mathfrak{J}}^b)$ and this product set is contained in $(\mathcal{O}_E + \mathfrak{P}_{\mathfrak{J}}^b)$. The 2×1 term of the product is easily seen to be contained in $\mathfrak{P}_{\mathfrak{J}}^a$. The 2×2 term is similar to 1×1. □

We denote by $H(a,b)$ the group of units of $\mathcal{H}(a,b)$. In the block form the group $H(a,b)$ is as follows:

$$\begin{pmatrix} J^0 & \mathcal{O}_E + \mathfrak{P}_{\mathfrak{J}}^b \\ \mathfrak{P}_{\mathfrak{J}}^a & J^0 \end{pmatrix}.$$

Lemma 6.2.14. *The representation λ_s of $H(a,b)\cap M$ extends to a representation of $H(a,b)$ such that $H(a,b)\cap U$ and $H(a,b)\cap \bar{U}$ are contained in the kernel of the extension.*

Proof. The proof is similar to that of 6.2.6. We note that $H(a,b)$ is equal to $H(2n,b)(H(a,b)\cap \bar{U})$. The representation λ_s extends to a representation of $H(2n,b)$ since $\mathrm{res}_{\mathrm{id}+\mathfrak{P}_{\mathfrak{J}}^{2n}}(\lambda_s) = \mathrm{id}$. Let u^+ and u^- be two elements of $H(a,b)\cap U$ and $H(a,b)\cap \bar{U}$ respectively and we take them in the block form as in lemma 6.2.6. We observe that $\lambda_s(u^- u^+ {u^-}^{-1}) = \psi_{[\alpha,B]}(\mathrm{id}+C)$. Now $B\in \mathcal{O}_E^\times + \mathfrak{P}_{\mathfrak{J}}^b$ hence $[\alpha,B]\in \mathfrak{P}_{\mathfrak{J}}^{-(2n-1)+b}$ and hence $\psi_{[\alpha,B]}(\mathrm{id}+C) = 1$ for all $C\in\mathfrak{P}_{\mathfrak{J}}^a$. This shows that $\lambda_s(u^- u^+ {u^-}^{-1}) = 1$. Hence the representation λ_s extends to $H(a,b)$ such that $H(a,b)\cap\bar{U}$ is in the kernel of the extended representation. □

Since $\mathfrak{P}_{\mathfrak{M}}^n \subset \mathfrak{P}_{\mathfrak{J}}^{2n-1}$ we get that $H_3 \subset H(2n-1,1)$. The induction $\mathrm{ind}_{H_3}^{H(2n-1,1)}(\lambda_s)$ has a unique complement of the representation λ_s (here λ_s is considered as a representation of $H(2n-1,1)$). Let $U(3,2n-1)$ be the complement of λ_s in $\mathrm{ind}_{H_3}^{H(2n-1,1)}(\lambda_s)$.

Lemma 6.2.15. *The irreducible sub-representations of*

$$\mathrm{ind}_{H(2n-1,1)}^{\mathrm{GL}_4(\mathcal{O}_F)}(U(3,2n-1))$$

are not typical representations.

Proof. The proof is similar to the proof of 6.1.2. We denote by H' the group

$$\begin{pmatrix} \mathrm{id}+\mathfrak{P}_{\mathfrak{J}}^n & \mathcal{O}_E + \mathfrak{P}_{\mathfrak{J}} \\ \mathfrak{P}_{\mathfrak{J}}^{2n-1} & \mathrm{id}+\mathfrak{P}_{\mathfrak{J}}^n \end{pmatrix}.$$

We observe that H' is a normal subgroup of $H(2n-1,1)$ and $H'H_3 = H(2n-1,1)$. Using Mackey decomposition we get that

$$\mathrm{res}_{H'}\,\mathrm{ind}_{H_3}^{H(2n-1,1)}(\mathrm{id}) \simeq \oplus_{j=1}^p \eta_j$$

where η_j are characters of H' which are trivial on $H'\cap H_3$. The quotient is given by $\mathfrak{P}_{\mathfrak{J}}^{2n-1}/\mathfrak{P}_{\mathfrak{M}}^n$. Now the group $H(2n-1,1)$ acts on these characters η_i and let $Z(\eta)$ be the $H(2n-1,1)$ stabilizer of η. Note that $Z(\eta) = (\mathcal{O}_E^\times \times \mathcal{O}_E^\times)H(2n-1,1)$. From Clifford theory we get that

$$\mathrm{ind}_{H_3}^{H(2n-1,1)}(\mathrm{id}) \simeq \oplus_{i\in\Lambda}\,\mathrm{ind}_{Z(\eta_i)}^{H(2n-1,1)}(U_{\eta_i})$$

where Λ is a set of representatives for the action of $H(2n-1,1)$ on the characters η_j for $1 \leq j \leq p$ and U_{η_i} is an irreducible representation of $Z(\eta_i)$. The action of $\mathcal{O}_E^\times \times \mathcal{O}_E^\times$ on the quotient $\mathfrak{P}_{\mathfrak{J}}^{2n-1}/\mathfrak{P}_{\mathfrak{M}}^n$ factors through $U_2(E)$ and if η is non-trivial character, from the arguments of **Case 3.3** in theorem 6.1.2 we get that modulo $\mathfrak{P}_E$ reduction of the group $Z(\eta)$ is of the form $\{(e,e) \mid e \in k_F^\times\}$. This shows that for $\#k_F > 3$ we can choose a character η of $k_F^\times$ such that the multi-sets $\{\lambda_1\eta, \lambda_2\eta^{-1}\}$ and $\{\lambda_1, \lambda_2\}$ are distinct. Moreover

$$\operatorname{ind}_{Z(\eta_i)}^{H(2n-1,1)}\{U_{\eta_i} \otimes (\lambda_1 \boxtimes \lambda_2)\} \simeq \operatorname{ind}_{Z(\eta_i)}^{H(2n-1,1)}\{U_{\eta_i} \otimes (\lambda_1\eta \boxtimes \lambda_2\eta^{-1})\}$$

for any $\eta \neq \mathrm{id}$. This shows that irreducible sub-representations of

$$\operatorname{ind}_{H(2n-1,1)}^{\mathrm{GL}_4(\mathcal{O}_F)}(U(3,2n-1))$$

are atypical representations. □

The above lemma shows that typical representations can only occur as sub-representations of $\operatorname{ind}_{H(2n-1,1)}^{\mathrm{GL}_4(\mathcal{O}_F)}(\lambda_s)$. We suppose that typical representations occur as sub-representations of $\operatorname{ind}_{H(a,b)}^{\mathrm{GL}_4(\mathcal{O}_F)}(\lambda_s)$ for some positive integer b such that $1 \leq b < n-1$ then we show that typical representations occur as sub-representation of $\operatorname{ind}_{H(a-1,b+1)}^{\mathrm{GL}_4(\mathcal{O}_F)}(\lambda_s)$. By Mackey decomposition we note that

$$\dim_{\mathbb{C}} \operatorname{Hom}(\operatorname{ind}_{H(a,b)}^{H(a-1,b)}(\lambda_s), \operatorname{ind}_{H(a-1,b+1)}^{H(a-1,b)}(\lambda_s)) = 1.$$

Let $I(a,b)$ be the non-trivial intertwining operator between $\operatorname{ind}_{H(a,b)}^{H(a-1,b)}(\lambda_s)$ and $\operatorname{ind}_{H(a-1,b+1)}^{H(a-1,b)}(\lambda_s)$.

Lemma 6.2.16. *The representation* $\operatorname{ind}_{H(a-1,b+1)}^{H(a-1,b)}(\lambda_s)$ *is irreducible.*

Lemma 6.2.17. *The irreducible sub-representations of* $\operatorname{ind}_{H(a-1,b)}^{\mathrm{GL}_4(\mathcal{O}_F)}(\ker(I(a,b)))$ *are not-typical representations.*

Proof of lemma 6.2.16 and 6.2.17. We apply the formalism developed in lemma 6.2.11 for the tuple $(H(a,b), H(a-1,b), H(a-1,b+1), \lambda_s)$. We have $\mathfrak{N}_1 = \mathcal{O}_E + \mathfrak{P}_{\mathfrak{J}}^b$ and $\mathfrak{N}_2 = \mathcal{O}_E + \mathfrak{P}_{\mathfrak{J}}^{b+1}$. In the language of Bushnell-Kutzko $\mathfrak{N}_1 = \mathfrak{N}_{-2n+b+1}(\beta,\mathfrak{J})$ and $\mathfrak{N}_2 = \mathfrak{N}_{-2n+b+2}(\beta,\mathfrak{J})$. We note that $\mathfrak{P}_2^* = \mathfrak{P}_{\mathfrak{J}}^{1-a} = \mathfrak{P}_{\mathfrak{J}}^{-2n+b+1}$ and $\mathfrak{P}_1^* = \mathfrak{P}_{\mathfrak{J}}^{-2n+b+2}$. The exact sequence 6.14 is given by [BK93, corollary 1.4.10] which says that the sequence

$$\frac{\mathfrak{N}_k(\beta,\mathfrak{J})}{\mathfrak{N}_{k+1}(\beta,\mathfrak{J})} \xrightarrow{[\alpha,\ ]} \frac{\mathfrak{P}_{\mathfrak{J}}^k}{\mathfrak{P}_{\mathfrak{J}}^{k+1}} \xrightarrow{s_{E/F}} \frac{\mathfrak{P}_E^k}{\mathfrak{P}_E^{k+1}}$$

is exact for all $k \geq k_0(\beta, \mathfrak{J})$ and in our context $k_0(\beta, \mathfrak{J}) = -2n+1$ and $k = -2n+b+1$. The irreducible sub-representations of $\ker(I(a,b))$ also occur in the representation

$$\mathrm{ind}_{H(a,b)}^{H(a-1,b)}(\lambda_1 \eta \boxtimes \lambda_2 \eta^{-1}).$$

for any tame character η of $\mathcal{O}_E^{\times}$. If $\# k_F > 3$ then we can choose a character η of $k_E^{\times} = k_F^{\times}$ such that the multi-sets types $\{\lambda_1, \lambda_2\}$ and $\{\lambda_1\eta, \lambda_2\eta^{-1}\}$ are distinct. If σ_1' and σ_2' are two supercuspidal representations containing $(J^0, \lambda_1\eta)$ and $(J^0, \lambda_2\eta^{-1})$ respectively then $s' = [M, \sigma_1' \boxtimes \sigma_2']$ and $s = [M, \sigma_1 \boxtimes \sigma_2]$ are different inertial classes and hence the irreducible sub-representations of $\mathrm{ind}_{H(a-1,b)}^{\mathrm{GL}_4(\mathcal{O}_F)}(\ker(I(a,b)))$ occur as sub-representations of $\mathrm{ind}_{H(a,b)}^{\mathrm{GL}_4(\mathcal{O}_F)}(\lambda_1\eta \boxtimes \lambda_2\eta^{-1})$ and hence are atypical. $\square$

This brings us to the final step of this section from the lemma 6.2.17 we conclude that typical representations occur as sub-representations of

$$\mathrm{ind}_{H(n+1,n-1)}^{\mathrm{GL}_4(\mathcal{O}_F)}(\lambda_s).$$

From lemma 6.2.4 we get that typical representations occur as sub-representations of $\mathrm{ind}_{J_s}^{\mathrm{GL}_4(\mathcal{O}_F)}(\lambda_s)$ where (J_s, λ_s) is the Bushnell-Kutzko type for the component s.

6.3 Completely distinct inertial classes

In this section we assume that $\# k_F > 3$ We fix Bushnell-Kutzko types (J_1^0, λ_1) and (J_2^0, λ_2) contained in the supercuspidal representations σ_1 and σ_2 respectively. Let $[\mathfrak{A}_1, n_1, 0, \beta_1]$ and $[\mathfrak{A}_2, n_2, 0, \beta_2]$ be two simple strata defining the types (J_1^0, λ_1) and (J_2^0, λ_2) respectively. Let $e_i = e(\mathfrak{A}_i | \mathcal{O}_F)$ for $i \in \{1, 2\}$. In this section we consider the case where σ_1 and σ_2 are completely distinct. **We will always assume that** $n_1/e_1 \geq n_2/e_2$. We denote by Λ_i the lattice sequence $\Lambda_i(r) = \mathcal{L}_i(-[-r])$ for all $r \in \mathbb{R}$ where $\mathcal{L}_i$ is the lattice chain defining the hereditary order $\mathfrak{A}_i$ and moreover we assume that $\mathcal{L}_i(0) = \mathcal{O}_F \oplus \mathcal{O}_F$ for $i \in \{1, 2\}$.

Let U and $\bar{U}$ be the unipotent radicals of P and the opposite parabolic subgroup of P with respect to M respectively. We denote by e the least common multiple of $e(\Lambda_1)$ and $e(\Lambda_2)$ respectively. Let l be the positive integer such that $l/e = \max\{n_1/e_1, n_2/e_2\}$. Let Λ be the direct sum of lattice sequences Λ_1 and Λ_2. The Bushnell-Kutzko type J_s satisfies the Iwahori decomposition with respect to the parabolic subgroup P and the Levi-subgroup M. The pair (J_s, λ_s) is characterised by the following properties

1. $J_s \cap U = u_0(\Lambda) \cap U$,

2. $J_s \cap \bar{U} = u_{l+1}(\Lambda) \cap \bar{U}$,

3. $J_s \cap M = J_1^0 \times J_2^0$.

4. The restriction of the representation λ_s to the subgroup $J_s \cap M$ is isomorphic to $\lambda_1 \boxtimes \lambda_2$ and $J_s \cap U$ and $J_s \cap \bar{U}$ are contained in the kernel of λ_s.

We refer to [BK99][Section 8, paragraph 8.3.1] for the construction of the pair (J_s, λ_s). We will explicitly compute $u_0(\Lambda) \cap U$ and $u_{l+1}(\Lambda) \cap \bar{U}$ in the following possibilities $e_1 = e_2 = 1$; $e_1 = 1, e_2 = 2$; $e_1 = 2, e_2 = 1$; and $e_1 = 2, e_2 = 2$.

We first consider the case $e_1 = e_2 = 1$. In this case

$$\Lambda(0) = \mathcal{O}_F \oplus \mathcal{O}_F \oplus \mathcal{O}_F \oplus \mathcal{O}_F$$

and $\Lambda(i+1) = \varpi_F \Lambda(i)$. Hence $u_0(\Lambda)$ is $\mathrm{GL}_4(\mathcal{O}_F)$ and $u_{l+1}(\Lambda)$ is the principal congruence subgroup of level $l+1$ inside $\mathrm{GL}_4(\mathcal{O}_F)$. This shows that $u_0(\Lambda) \cap U = U(\mathcal{O}_F)$ and $u_{l+1}(\Lambda) \cap \bar{U} = \bar{U}(\mathfrak{P}_F^{l+1})$. Moreover $N = l$ Hence we observe that in this case $J_s = P^0(N+1)$. Hence the theorem 6.1.2 completes classification of typical representations in the case where $e_1 = e_2 = 1$ and σ_1 and σ_2 are completely distinct.

We will now consider the case $e_1 = 1, e_2 = 2$. In this situation

$$\Lambda(0) = \mathcal{O}_F \oplus \mathcal{O}_F \oplus \mathcal{O}_F \oplus \mathcal{O}_F$$
$$\Lambda(1) = \Lambda_1(1/2) \oplus \Lambda_2(1) = \mathfrak{P}_F \oplus \mathfrak{P}_F \oplus \mathcal{O}_F \oplus \mathfrak{P}_F$$

and $\Lambda(i+2) = \varpi_F \Lambda(i)$. Let $\mathfrak{n}$ and $\bar{\mathfrak{n}}$ be the upper and lower nilpotent matrices of the type $(2,2)$ i.e the Lie algebras of U and $\bar{U}$ respectively. Now $u_i(\Lambda) \cap U = 1 + (a_i(\Lambda) \cap \mathfrak{n})$ and $u_i(\Lambda) \cap \bar{U} = 1 + (a_i(\Lambda) \cap \bar{\mathfrak{n}})$ ($a_i(\Lambda)$ is defined in the section 5.1). We note that $a_0 \cap \mathfrak{n}$ is the set

$$\{x \in M_4(F) \cap \mathfrak{n} \mid x\Lambda(i) \subset \Lambda(i) \forall i \in \mathbb{Z}\}$$

In our case it is given by the set

$$\begin{pmatrix} 0 & 0 & \mathfrak{P}_F & \mathcal{O}_F \\ 0 & 0 & \mathfrak{P}_F & \mathcal{O}_F \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Similarly the groups $a_0(\Lambda) \cap \bar{\mathfrak{n}}$ and $a_1(\Lambda) \cap \bar{\mathfrak{n}}$ are given by

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \mathcal{O}_F & \mathcal{O}_F & 0 & 0 \\ \mathcal{O}_F & \mathcal{O}_F & 0 & 0 \end{pmatrix} \text{ and } \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \mathcal{O}_F & \mathcal{O}_F & 0 & 0 \\ \mathfrak{P}_F & \mathfrak{P}_F & 0 & 0 \end{pmatrix}$$

respectively. Let $l+1 = 2l'+r$ with $r \in \{0,1\}$ then $a_{l+1}(\Lambda) \cap \bar{\mathfrak{n}} = \varpi_F^{l'}(a_r(\Lambda) \cap \bar{\mathfrak{n}})$. We recall that $N = \max\{n_1, [(n_2+1)/2]\}$ and $l/2 = \max\{n_1, n_2/2\} = n_1$ since we assume that $n_1 \geq n_2/2$. Since n_1 is a positive integer, $n_1 \geq n_2/2$ if and only if $n_1 \geq [(n_2+1)/2]$. In this case $N = n_1$. Hence we deduce that $2N = l$. From this we get that $a_{l+1}(\Lambda) \cap \bar{\mathfrak{n}} = \varpi_F^N(a_1(\Lambda) \cap \bar{\mathfrak{n}})$. The first observation is that the dimensions of the representations

$$\mathrm{ind}_{P^0(N+1)}^{\mathrm{GL}_4(\mathcal{O}_F)}(\lambda_1 \boxtimes \lambda_2) \quad \text{and} \quad \mathrm{ind}_{J_s}^{\mathrm{GL}_4(\mathcal{O}_F)}(\lambda_1 \boxtimes \lambda_2)$$

are the same and moreover the intertwining of the second representation from the lemma [BK98][lemma 11.5] is bounded by the cardinality of $N_{\mathrm{GL}_4(F)}(s)/M$. Since the representations σ_1 and σ_2 are not inertially equivalent we get that the cardinality of $N_{\mathrm{GL}_4(F)}(s)/M$ is one. Hence the representation

$$\mathrm{ind}_{J_s}^{\mathrm{GL}_4(\mathcal{O}_F)}(\lambda_1 \boxtimes \lambda_2)$$

is irreducible. Now at least one typical representation must be contained in $i_P^{\mathrm{GL}_4(F)}(\sigma_1 \boxtimes \sigma_2)$ and lemma 2.2.4, theorem 6.1.2 gives the inclusion of the above representation in

$$\mathrm{ind}_{P^0(N+1)}^{\mathrm{GL}_4(\mathcal{O}_F)}(\lambda_1 \boxtimes \lambda_2)$$

Hence we have the isomorphism

$$\mathrm{ind}_{J_s}^{\mathrm{GL}_4(\mathcal{O}_F)}(\lambda_1 \boxtimes \lambda_2) \simeq \mathrm{ind}_{P^0(N+1)}^{\mathrm{GL}_4(\mathcal{O}_F)}(\lambda_1 \boxtimes \lambda_2). \tag{6.19}$$

Let us consider the case where $e_1 = 2$ and $e_2 = 1$. In this case $\Lambda(i+2) = \varpi_F \Lambda(i)$ and

$$\begin{aligned}
\Lambda(0) &= \mathcal{O}_F \oplus \mathcal{O}_F \oplus \mathcal{O}_F \oplus \mathcal{O}_F, \\
\Lambda(1) &= \Lambda_1(1) \oplus \Lambda_2(1/2) = \mathcal{O}_F \oplus \mathfrak{P}_F \oplus \mathfrak{P}_F \oplus \mathfrak{P}_F.
\end{aligned}$$

The group $a_0(\Lambda) \cap \mathfrak{n}$ is given by

$$\begin{pmatrix} 0 & 0 & \mathcal{O}_F & \mathcal{O}_F \\ 0 & 0 & \mathcal{O}_F & \mathcal{O}_F \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

The group $a_0(\Lambda) \cap \bar{\mathfrak{n}}$ and $a_1(\Lambda) \cap \bar{\mathfrak{n}}$ is given by

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \mathfrak{P}_F & \mathcal{O}_F & 0 & 0 \\ \mathfrak{P}_F & \mathcal{O}_F & 0 & 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \mathfrak{P}_F & \mathfrak{P}_F & 0 & 0 \\ \mathfrak{P}_F & \mathfrak{P}_F & 0 & 0 \end{pmatrix}.$$

In this situation $n_1/2 \geq n_2$ and hence $l/2 = \max\{n_1/2, n_2\} = n_1/2$. Now consider the positive integer $N = \max\{[(n_1+1)/2], n_2\}$. Since $n_1/2 \geq n_2$ and $[(n_1+1)/2] \geq n_1/2$ we get that $N = [(n_1+1)/2]$. Here we can use the fact that

n_1 is odd and we have $a_{l+1}(\Lambda) \cap \bar{\mathfrak{n}} = \varpi_F^{(n_1+1)/2}(a_0(\Lambda) \cap \bar{\mathfrak{n}}) = \varpi_F^N(a_0(\Lambda) \cap \bar{\mathfrak{n}})$. Observe that $P^0(N+1) \subset J_s$.

The representation λ_s occurs with multiplicity one in the representation $\mathrm{ind}_{P^0(N+1)}^{J_s}(\lambda_s)$. We denote by $U_N^0(\lambda_s)$ the unique complement of λ_s in the representation $\mathrm{ind}_{P^0(N+1)}^{J_s}(\lambda_s)$. We denote by $U_N(\lambda_s)$ the representation

$$\mathrm{ind}_{J_s}^{\mathrm{GL}_4(\mathcal{O}_F)}(U_N^0(\lambda_s)).$$

Lemma 6.3.1. *The irreducible sub-representations of $U_N(\lambda_s)$ are atypical.*

Proof. The proof of this lemma is similar to 5.3.2. The first step is to split the representation

$$\mathrm{ind}_{P^0(N+1)}^{J_s}(\mathrm{id}).$$

To begin with we will show that the group $U(\mathcal{O}_F)$ acts trivially on the above representation. Let u^+ be an element of $U(\mathcal{O}_F)$ and u^- be an element of $J_s \cap \bar{U}$. We denote u^+ and u^- in their respective block form as

$$\begin{pmatrix} 1_2 & U^+ \\ 0 & 1_2 \end{pmatrix} \text{ and } \begin{pmatrix} 1_2 & 0 \\ U^- & 1_2 \end{pmatrix}.$$

Now the conjugation $u^-u^+(u^-)^{-1}$ is of the form

$$\begin{pmatrix} 1_2 - U^+U^- & U^+ \\ U^-U^+U^- & 1_2 + U^-U^+ \end{pmatrix}.$$

We observe that $U^-U^+U^- \in \varpi^{N+1}M_2(\mathcal{O}_F)$. This shows that the conjugation $u^-u^+(u^-)^{-1}$ lies in the group $P^0(N+1)$. Hence the group $U(\mathcal{O}_F)$ acts trivially on

$$\mathrm{ind}_{P^0(N+1)}^{J_s}(\mathrm{id}). \tag{6.20}$$

From the Iwahori decomposition of the group J_s we get that J_s is equal to $(J_s \cap \bar{P})P^0(N+1)$. Hence we get that

$$\mathrm{res}_{J_s \cap \bar{P}_I} \mathrm{ind}_{P^0(N+1)}^{J_s}(\mathrm{id}) \simeq \mathrm{ind}_{P^0(N+1)\cap \bar{P}}^{J_s \cap \bar{P}}(\mathrm{id}).$$

Note that $J_s \cap \bar{P}$ is a semi-direct product of the groups $(J_s \cap M)$ and $(J_s \cap \bar{U})$. Let η_k for $1 \leq k \leq t$ (we mean counting them with their multiplicity, but in our case the multiplicity is one) be all the characters of the group $J_s \cap \bar{U}$ which are trivial on the group $P^0(N+1) \cap \bar{U}$. The group $J_s \cap \bar{P}$ acts on these characters and let $\{\eta_{k_p}\}$ be a set of representatives for the orbits under this action. We denote by $Z(\eta_{k_p})$ the $J_s \cap \bar{P}$ stabiliser of the character η_{k_p}. Now Clifford theory gives the decomposition

$$\mathrm{ind}_{P^0(N+1)\cap \bar{P}}^{J_s \cap \bar{P}}(\mathrm{id}) \simeq \bigoplus_{\eta_{k_p}} \mathrm{ind}_{Z(\eta_{k_p})}^{J_s \cap \bar{P}}(U_{\eta_{k_p}})$$

where $U_{\eta_{k_p}}$ is an irreducible representation of $Z(\eta_{k_p})$. We note that the character id occurs with a multiplicity one in the list of characters η_k.

The representation $U_{\eta_{k_p}}$ is the isotypic component of the character η_{k_p} in the representation

$$\mathrm{ind}_{P^0(N+1)\cap\bar{P}}^{J_s\cap\bar{P}}(\mathrm{id}).$$

which naturally has the action of $Z(\eta_{k_p})$. Now if K_s is the kernel of the representation (6.20) then $K_s \cap Z(\eta_{k_p})$ acts trivially on $U_{\eta_{k_p}}$. Hence we can extend the representation $U_{\eta_{k_p}}$ to the group $Z(\eta_{k_p})K_s$ such that K_s acts trivially on the extended representation. Now consider the representation

$$\pi = \mathrm{ind}_{Z(\eta_{k_p})K_s}^{J_s} U_{\eta_{k_p}}.$$

Note that $K_s \cap \bar{P}$ is contained in the group $Z(\eta_{k_p}) \cap \bar{P}$ and moreover $U(\mathcal{O}_F)$ is contained in K_s hence $J_s = (J_s \cap \bar{P})Z(\eta_{k_p})K_s$. Using Mackey decomposition we have

$$\mathrm{res}_{J_s\cap\bar{P}}\, \mathrm{ind}_{Z(\eta_{k_p})K_s}^{J_s} U_{\eta_{k_p}} \simeq \mathrm{ind}_{Z(\eta_{k_p})K_s\cap(J_s\cap\bar{P})}^{J_s\cap\bar{P}}(U_{\eta_{k_p}}) \simeq \mathrm{ind}_{Z(\eta_{k_p})}^{J_s\cap\bar{P}}(U_{\eta_{k_p}}).$$

We hence obtain

$$\mathrm{ind}_{P^0(N+1)}^{J_s}(\mathrm{id}) \simeq \bigoplus_{\eta_{k_p}} \mathrm{ind}_{Z(\eta_{k_p})K_s}^{J_s} U_{\eta_{k_p}}. \tag{6.21}$$

Now using the decomposition (6.21) we get that the decomposition

$$\mathrm{ind}_{P^0(N+1)}^{\mathrm{GL}_4(\mathcal{O}_F)}(\lambda_1 \boxtimes \lambda_2) \simeq \bigoplus_{\eta_{k_p}} \mathrm{ind}_{Z(\eta_{k_p})K_s}^{\mathrm{GL}_4(\mathcal{O}_F)}\{U_{\eta_{k_p}} \otimes (\lambda_1 \boxtimes \lambda_2)\}.$$

Note that the character id occurs with multiplicity one among the characters η_k and the fact that $Z(\mathrm{id})K_s = (J_s \cap \bar{P}_I)K_s = J_s$ implies the following isomorphism

$$\mathrm{ind}_{P^0(N+1)}^{\mathrm{GL}_4(\mathcal{O}_F)}(\lambda_1 \boxtimes \lambda_2) \simeq \mathrm{ind}_{J_s}^{\mathrm{GL}_4(\mathcal{O}_F)}(\lambda_1 \boxtimes \lambda_2) \oplus \bigoplus_{\eta_{k_p}\neq \mathrm{id}} \mathrm{ind}_{Z(\eta_{k_p})K_s}^{\mathrm{GL}_4(\mathcal{O}_F)}\{U_{\eta_{k_p}} \otimes (\lambda_1 \boxtimes \lambda_2)\}. \tag{6.22}$$

Let Γ be an irreducible sub-representations of

$$\mathrm{ind}_{Z(\eta_{k_p})K_s}^{\mathrm{GL}_4(\mathcal{O}_F)}\{U_{\eta_{k_p}} \otimes (\lambda_1 \boxtimes \lambda_2)\}.$$

From the reasoning given in **Case 3.2** of theorem 6.1.2, for $\#k_F > 3$, we can find two types (J_1^0, λ_1') and (J_2^0, λ_2') such that $[M, \sigma_1 \boxtimes \sigma_2] \neq [M, \sigma_1 \boxtimes \sigma_2]$ where σ_1' and σ_2' are two supercuspidal representations containing (J_1^0, λ_1') and (J_2^0, λ_2') respectively,

$$\mathrm{res}_{Z(\eta_{k_p})}(\lambda_1 \boxtimes \lambda_2) \simeq \mathrm{res}_{Z(\eta_{k_p})}(\lambda_1' \boxtimes \lambda_2').$$

This shows that the representation Γ is not a typical representation. $\square$

Now we look at our last possibility $e_1 = e_2 = 2$. In this case $\Lambda(i+2) = \varpi_F \Lambda(i)$, $\Lambda(0) = \mathcal{O}_F \oplus \mathcal{O}_F \oplus \mathcal{O}_F \oplus \mathcal{O}_F$ and $\Lambda(1) = \mathcal{O}_F \oplus \mathfrak{P}_F \oplus \mathcal{O}_F \oplus \mathfrak{P}_F$. The group $a_0(\Lambda) \cap \mathfrak{n}$ is given by

$$\begin{pmatrix} 0 & 0 & \mathcal{O}_F & \mathcal{O}_F \\ 0 & 0 & \mathfrak{P}_F & \mathcal{O}_F \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

The groups $a_0(\Lambda) \cap \bar{\mathfrak{n}}$ and $a_1(\Lambda) \cap \bar{\mathfrak{n}}$ are given by the groups

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \mathcal{O}_F & \mathcal{O}_F & 0 & 0 \\ \mathfrak{P}_F & \mathcal{O}_F & 0 & 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \mathfrak{P}_F & \mathcal{O}_F & 0 & 0 \\ \mathfrak{P}_F & \mathfrak{P}_F & 0 & 0 \end{pmatrix}$$

respectively. We have $l/2 = \max\{n_1/2, n_2/2\} = n_1/2$ and hence $l = n_1$. At the same time $N = \max\{[(n_1+1)/2], [(n_2+1)/2]\} = [(n_1+1)/2]$. We use the fact that n_1 is odd and get that $a_{l+1}(\Lambda) \cap \bar{\mathfrak{n}} = \varpi_F^{(n_1+1)/2}(a_0(\Lambda) \cap \bar{\mathfrak{n}}) = \varpi_F^N(a_0(\Lambda) \cap \bar{\mathfrak{n}})$.

Let H_1 be the group

$$\begin{pmatrix} J_1^0 & \varpi_F M_2(\mathcal{O}_F) \\ \varpi_F^N M_2(\mathcal{O}_F) & J_2^0 \end{pmatrix}.$$

The group H_1 is contained in the group $P^0(N)$.

Lemma 6.3.2. *The representation λ_s of $J_1^0 \times J_2^0$ extends to a representation of H_1 such that $H_1 \cap U$ and $H_1 \cap \bar{U}$ are contained in the kernel of this extension.*

Proof. The representation $\lambda_s = \lambda_1 \boxtimes \lambda_2$ extends to the on the group $P^0(N+1)$ such that $P^0(N+1) \cap U$ and $P^0(N+1) \cap \bar{U}$ are contained in the kernel of the extension. Hence the representation λ_s extends to the group $H_1 \cap P^0(N+1)$. We note that $H_1 = (P^0(N) \cap \bar{U})(H_1 \cap P^0(N+1))$. Let u^- be an element of $P^0(N) \cap \bar{U}$. To prove the lemma it is enough to verify that $(u^-)^{-1}u^+u^-$ belongs to $H_1 \cap P^0(N+1)$ and $\lambda_s((u^-)^{-1}u^+u^-) = 1$ for all $u^- \in H_1 \cap \bar{U}$ and $u^+ \in H_1 \cap U$ respectively.

We write u^- and u^+ in the block form as

$$\begin{pmatrix} 1_2 & 0 \\ U^- & 1_2 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1_2 & U^+ \\ 0 & 1_2 \end{pmatrix}$$

respectively and $U^- \in \varpi_F^N M_2(\mathcal{O}_F)$ and $U^+ \in \varpi_F M_2(\mathcal{O}_F)$. The conjugation $(u^-)^{-1}u^+u^-$ in the block form is given by

$$\begin{pmatrix} 1_2 - U^+U^- & U^+ \\ -U^-U^+U^- & U^-U^+ + 1_2 \end{pmatrix}.$$

Since $U^-U^+U^- \in \varpi_F^{2N+1}M_2(\mathcal{O}_F)$ we get that the above matrix belongs to $H_1 \cap P^0(N+1)$. The element $1_2 + U^-U^+$ and $1_2 - U^+U^-$ are contained in the kernel of λ_2 and λ_1 respectively. Hence $\lambda_s((u^-)^{-1}u^+u^-) = 1$. $\square$

From the observation $P^0(N) = P^0(N+1)H_1$ we get that

$$\dim_{\mathbb{C}} \operatorname{Hom}_{P^0(N)}(\operatorname{ind}_{P^0(N+1)}^{P^0(N)}(\lambda_s), \operatorname{ind}_{H_1}^{P^0(N)}(\lambda_s)) = 1.$$

Lemma 6.3.3. *The representation* $\operatorname{ind}_{H_1}^{P^0(N)}(\lambda_s)$ *is an irreducible representation of* $P^0(N)$.

Proof. We prove the lemma by showing that the support of the intertwining of the representation considered in the lemma is contained in the double coset containing identity. The double coset representatives for $H_1 \backslash P^0(N)/H_1$ can be chosen from the group $P^0(N) \cap U$. Let u^+ be an element from the group $P^0(N) \cap U$. Let the following matrix be the block from of u^+:

$$\begin{pmatrix} 1_2 & U^+ \\ 0 & 1_2 \end{pmatrix}$$

where $U^+ \in M_2(\mathcal{O}_F)$. Let T be a non-zero operator in the space

$$\operatorname{Hom}_{H_1 \cap (H_1)^{u^+}}(\lambda_s, (\lambda_s)^{u^+}).$$

Now T must satisfy the relation

$$T(\begin{pmatrix} \mathrm{id} & 0 \\ C & \mathrm{id} \end{pmatrix} v) = \begin{pmatrix} \mathrm{id} + U^+C & -U^+CU^+ \\ C & -CU^+ + \mathrm{id} \end{pmatrix} T(v).$$

for all $v \in \operatorname{ind}_{H_1}^{P^0(N)}(\lambda_s)$ and $C \in \varpi_F^N M_2(\mathcal{O}_F)$. This implies that the character $\psi_{\beta_1 U^+ - U^+ \beta_2}(1 + C) = 1$. Since characteristic polynomials of β_1 and β_2 are relatively prime we get that $U^+ \in \varpi_F M_2(\mathcal{O}_F)$ (see [BK99][lemma 4 p.72]). Hence the intertwining is supported only on the double coset containing identity. This shows the lemma. $\square$

We note that $\operatorname{ind}_{P^0(N+1)}^{P^0(N)}(\lambda_s)$ and $\operatorname{ind}_{H_1}^{P^0(N)}(\lambda_s)$ are of the same dimension. The above lemma together with the transitivity of induction we get that

$$\operatorname{ind}_{P^0(N+1)}^{\mathrm{GL}_4(\mathcal{O}_F)}(\lambda_s) \simeq \operatorname{ind}_{H_1}^{\mathrm{GL}_4(\mathcal{O}_F)}(\lambda_s).$$

Let H_2 be the group

$$\begin{pmatrix} J_1^0 & \mathfrak{P}_{\mathfrak{A}_1} \\ \varpi_F^N M_2(\mathcal{O}_F) & J_2^0 \end{pmatrix}.$$

(Note that $\mathfrak{A}_1$ is a hereditary order defined by the lattice sequence Λ_1 with periodicity 2 and $\Lambda_1(0) = \mathcal{O}_F \oplus \mathcal{O}_F$ and $\Lambda_1(1) = \mathcal{O}_F \oplus \mathfrak{P}_F$.) We observe that

$H_1 \subset H_2$ and moreover $H_1 \cap \bar{U} = H_2 \cap \bar{U}$ and $H_1 \cap M = H_2 \cap M$. We will also need another group H_3 given by

$$\begin{pmatrix} J_1^0 & \mathfrak{P}_{\mathfrak{A}_1} \\ \mathfrak{P}_{\mathfrak{A}_1}^{2N-1} & J_2^0 \end{pmatrix}.$$

The representation λ_s of $H_3 \cap M = J_1^0 \times J_2^0$ extends to a representation of H_3 such that λ_s extends to a representation of H_3 such that $H_3 \cap U$ and $H_3 \cap \bar{U}$ are contained in the kernel of the extension the proof is similar to 6.3.2, the only important fact is that kernels of λ_1 and λ_2 contain the group $U^{2N}(\mathfrak{A}_1)$. Since $\varpi_F^N M_2(\mathcal{O}_F) \subset \mathfrak{P}_{\mathfrak{A}_1}^{2N-1}$, we get that $H_2 \subset H_3$. From this we also get that the representation λ_s of $H_2 \cap M$ extends to a representation of H_2 such that $H_2 \cap U$ and $H_2 \cap \bar{U}$ are contained in the kernel of this extension.

It is by now a standard practice (see lemma 6.3.1) to decompose $\mathrm{ind}_{H_i}^{H_{i+1}}(\mathrm{id})$ as a direct sum

$$\mathrm{id} \oplus \bigoplus_j \mathrm{ind}_{Z_j^i}^{H_{i+1}}(U_j)$$

where the mod $\mathfrak{P}_F$ reduction of $Z_j^i \cap M$ is contained in the $M(k_F)$ stabiliser of a non-zero matrix A in $M_{2\times 2}(k_F)$ for $i \in \{1, 2\}$. From the arguments in the **Case 3.3** of theorem 6.1.2, for $\#k_F > 3$, we can find two types (J_1^0, λ_1') and (J_2^0, λ_2') such that $[M, \sigma_1 \boxtimes \sigma_2] \neq [M, \sigma_1 \boxtimes \sigma_2]$ where σ_1' and σ_2' are two supercuspidal representations containing (J_1^0, λ_1') and (J_2^0, λ_2') respectively,

$$\mathrm{res}_{Z_j^i}(\lambda_1 \boxtimes \lambda_2) \simeq \mathrm{res}_{Z_j^i}(\lambda_1' \boxtimes \lambda_2').$$

This shows that irreducible sub-representations of

$$\mathrm{ind}_{Z_j^i}^{\mathrm{GL}_4(\mathcal{O}_F)}\{\lambda_s \otimes U_j\}$$

are atypical for $\#k_F > 3$ and $i \in \{1, 2\}$.

Now observe that the dimensions of $\mathrm{ind}_{J_s}^{\mathrm{GL}_4(\mathcal{O}_F)}(\lambda_s)$ and $\mathrm{ind}_{H_3}^{\mathrm{GL}_4(\mathcal{O}_F)}(\lambda_s)$ are the same. Note that the type (J_s, λ_s) occurs in the smooth representation $i_P^{\mathrm{GL}_4(F)}(\sigma_1 \boxtimes \sigma_2)$. This shows that at least one irreducible sub-representation of

$$\mathrm{ind}_{J_s}^{\mathrm{GL}_4(\mathcal{O}_F)}(\lambda_s) \tag{6.23}$$

must be contained in $i_P^{\mathrm{GL}_4(F)}(\sigma_1 \boxtimes \sigma_2)$. The representation (6.23) is irreducible and typical. From our results so far typical representations for $s = [M, \sigma_1 \boxtimes \sigma_2]$ must occur as sub-representations of $\mathrm{ind}_{H_3}^{\mathrm{GL}_4(\mathcal{O}_F)}(\lambda_s)$. Hence we have an isomorphism

$$\mathrm{ind}_{J_s}^{\mathrm{GL}_4(\mathcal{O}_F)}(\lambda_s) \simeq \mathrm{ind}_{H_3}^{\mathrm{GL}_4(\mathcal{O}_F)}(\lambda_s).$$

This concludes the classification of typical representations in this case.

The results of this chapter can be collected in the following theorem.

Theorem 6.3.4. *Let $s = [M, \sigma_1 \boxtimes \sigma_2]$ be an inertial class such that σ_1 and σ_2 are either completely distinct or homogenous. Let $\# k_F > 3$. If Γ is a typical representation for the component s then Γ is a sub-representation of*

$$\mathrm{ind}_{J_s}^{\mathrm{GL}_4(\mathcal{O}_F)}(\lambda_s)$$

where (J_s, λ_s) is the Bushnell-Kutzko semi-simple type for s and moreover

$$\dim_{\mathbb{C}} \mathrm{Hom}_{\mathrm{GL}_4(\mathcal{O}_F)}(\Gamma, \mathrm{ind}_{J_s}^{\mathrm{GL}_4(\mathcal{O}_F)}(\lambda_s)) = \dim_{\mathbb{C}} \mathrm{Hom}_{\mathrm{GL}_4(\mathcal{O}_F)}(\Gamma, i_P^{\mathrm{GL}_4(F)}(\sigma_1 \boxtimes \sigma_2))$$

where P is any parabolic subgroup containing M as its Levi-subgroup.

Acknowledgements

I express my deepest gratitude to my thesis supervisor Guy Henniart for his guidance during my masters and doctorate years. I learnt a great deal of mathematics, about research and about writing mathematics. I learned the theory of smooth representations of p-adic reductive groups essentially from him. I thank him for the encouragement and support through difficult times in graduate school.

I spent almost an year at Leiden university and had numerous discussions on various aspects of modular forms, Galois representations and geometry among other things with Bas Edixhoven. I am deeply influenced by those discussions we had every week. I would like to thank my thesis co-supervisor Bas Edixhoven for these stimulating discussions and for his encouragement. I also thank him for helping me with the French translation.

I would like to thank Shaun Stevens for his interest in my work. It is indeed his comments on my work on GL_3 encouraged me to think about the certain difficult cases of GL_4. I would like to thank Corinne Blondel for her detailed comments on the thesis and some simplifications she suggested. I thank David Helm for his report on the thesis.

I would like to thank Jyoti Prakash Saha, Ramon M. Nunes, Shaunak Deo, Vineeth Ch and Sourav Ghosh for the mathematical discussions we had. I am very grateful for the libraries of Orsay and Indian Statistical Institute, Bangalore. I thank all my teachers in Indian statistical Institute, Bangalore, University of Padova and Université Paris sud. Last but not the least I thank my family amma, nanna, satish and tanu for keeping up with me.

Abstract

In this thesis we classify typical representations for certain non-cuspidal Bernstein components of GL_n over a non-Archimedean local field. Following the work of Henniart in the case of GL_2 and Paskunas for the cuspidal Bernstein components, we classify typical representations for Bernstein components of level-zero for GL_n for $n \geq 3$, principal series components, components with Levi subgroup of the form $(n, 1)$ for $n > 1$ and certain components with Levi subgroup of the form $(2, 2)$.

Each of the above component is treated in a separate chapter. The classification uses the theory of types developed by Bushnell and Kutzko in a significant way. We will give the classification in terms of Bushnell-Kutzko types for a given inertial class.

Samenvatting

In deze scriptie classificeren wij typische representaties voor bepaalde niet-cuspidale Bernstein componenten van GL_n over een niet Archemedisch lokaal lichaam. Gebruik makend van het werk van Henniart in het geval van GL_2 voor de cuspidale Bernstein componenten, classificeren wij representaties voor Bernstein componenten van niveau nul voor GL_n voor $n \geq 3$, voor principal series components, voor componenten met Levi ondergroep van de vorm $(n, 1)$ met $n > 1$ en sommige componenten met Levi ondergroep van de vorm $(2, 2)$.

Alle bovenstaande componenten worden in hun eigen hoofdstuk behandeld. De classificatie berust zwaar op de theorie van typen ontwikkeld door Bushnell en Kutzko en deze zal ook gegeven worden in de termen van de Bushnell-Kutzko typen behorend bij een gegeven inertie klasse.

Résumé

Dans cette thèse, nous classifions les représentations typiques pour certaines composantes de Bernstein de GL_n sur un corps localement compact non Archimédien. Suite aux travaux de Henniart dans le cas de GL_2 et de Paskunas pour les composantes de Bernstein cuspidales, nous classifions les représentations typiques pour les composants de Bernstein de niveau zéro pour $n \geq 3$, les composantes de séries principales, les composantes dont le sous-groupe de Levi est de forme $(n, 1)$ pour $n > 1$ et certaines composantes dont le sous-groupe de Levi est de la forme $(2, 2)$.

Chacune des composantes ci-dessus est traitée dans un chapitre distinct. La classication utilise d'une manière significative la théorie des types développée par Bushnell-Kutzko, et elle est établie en termes de tels types.

Curriculum Vitae

Santosh Nadimpalli was born on 4^{th}-January 1990 in Vijayawada, India. He attended Visakha valley school in Visakhapatnam until his 10^{th} class. He completed his intermediate education in Hyderabad aiming to be an engineer. He got admission in Indian Statistical Institute to pursue bachelors of mathematics in 2007.

In 2010 he was offered Algant Masters scholarship to study in University of Padova and Universié Paris Sud. He spent the first year of masters in Padova $(2010 - 2011)$ and the second year $(2011 - 2012)$ in Orsay. As a second year masters student he approached Prof. Guy Henniart for Masters thesis. He wrote his masters thesis with Henniart and completed his masters in 2012.

With Algant Doctorate scholarship, in 2012 he began his Phd jointly hosted by University of Leiden and Université de Paris Sud. Following the suggestion of Henniart he is working on the topic of typical representations for the Phd.

Bibliography

[BH06] Colin J. Bushnell and Guy Henniart, *The local Langlands conjecture for* GL(2), Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], vol. 335, Springer-Verlag, Berlin, 2006. MR 2234120 (2007m:22013)

[BH13] ______, *Intertwining of simple characters in* GL(n), Int. Math. Res. Not. IMRN (2013), no. 17, 3977–3987. MR 3096916

[BK93] Colin J. Bushnell and Philip C. Kutzko, *The admissible dual of* GL(N) *via compact open subgroups*, Annals of Mathematics Studies, vol. 129, Princeton University Press, Princeton, NJ, 1993. MR 1204652 (94h:22007)

[BK98] ______, *Smooth representations of reductive p-adic groups: structure theory via types*, Proc. London Math. Soc. (3) **77** (1998), no. 3, 582–634. MR 1643417 (2000c:22014)

[BK99] ______, *Semisimple types in* GL_n, Compositio Math. **119** (1999), no. 1, 53–97. MR 1711578 (2000i:20072)

[Blo06] Corinne Blondel, *Propagation de paires couvrantes dans les groupes symplectiques*, Represent. Theory **10** (2006), 399–434 (electronic). MR 2266698 (2007m:22012)

[BM02] Christophe Breuil and Ariane Mézard, *Multiplicités modulaires et représentations de* $\mathrm{GL}_2(\mathbf{Z}_p)$ *et de* $\mathrm{Gal}(\overline{\mathbf{Q}}_p/\mathbf{Q}_p)$ *en* $l = p$, Duke Math. J. **115** (2002), no. 2, 205–310, With an appendix by Guy Henniart. MR 1944572 (2004i:11052)

[BZ76] I. N. Bernšteĭn and A. V. Zelevinskiĭ, *Representations of the group $GL(n, F)$, where F is a local non-Archimedean field*, Uspehi Mat. Nauk **31** (1976), no. 3(189), 5–70. MR 0425030 (54 #12988)

[BZ77] I. N. Bernstein and A. V. Zelevinsky, *Induced representations of reductive p-adic groups. I*, Ann. Sci. École Norm. Sup. (4) **10** (1977), no. 4, 441–472. MR 0579172 (58 #28310)

[Cas73] William Casselman, *The restriction of a representation of* $\mathrm{GL}_2(k)$ *to* $\mathrm{GL}_2(\mathfrak{o})$, Math. Ann. **206** (1973), 311–318. MR 0338274 (49 #3040)

[EG14] Matthew Emerton and Toby Gee, *A geometric perspective on the Breuil-Mézard conjecture*, J. Inst. Math. Jussieu **13** (2014), no. 1, 183–223. MR 3134019

[Hen00] Guy Henniart, *Une preuve simple des conjectures de Langlands pour* GL(n) *sur un corps p-adique*, Invent. Math. **139** (2000), no. 2, 439–455. MR 1738446 (2001e:11052)

[How73] Roger E. Howe, *On the principal series of* Gl_n *over p-adic fields*, Trans. Amer. Math. Soc. **177** (1973), 275–286. MR 0327982 (48 #6324)

[HT01] Michael Harris and Richard Taylor, *The geometry and cohomology of some simple Shimura varieties*, Annals of Mathematics Studies, vol. 151, Princeton University Press, Princeton, NJ, 2001, With an appendix by Vladimir G. Berkovich. MR 1876802 (2002m:11050)

[LRS93] G. Laumon, M. Rapoport, and U. Stuhler, *D-elliptic sheaves and the langlands correspondence*, Inventiones mathematicae **113** (1993), no. 1, 217–338 (English).

[Pas05] Vytautas Paskunas, *Unicity of types for supercuspidal representations of* GL_N, Proc. London Math. Soc. (3) **91** (2005), no. 3, 623–654. MR 2180458 (2007b:22018)

[Ren10] David Renard, *Représentations des groupes réductifs p-adiques*, Cours Spécialisés [Specialized Courses], vol. 17, Société Mathématique de France, Paris, 2010. MR 2567785 (2011d:22019)

[Roc98] Alan Roche, *Types and Hecke algebras for principal series representations of split reductive p-adic groups*, Ann. Sci. École Norm. Sup. (4) **31** (1998), no. 3, 361–413. MR 1621409 (99d:22028)

[SZ99] P. Schneider and E.-W. Zink, *K-types for the tempered components of a p-adic general linear group*, J. Reine Angew. Math. **517** (1999), 161–208, With an appendix by Schneider and U. Stuhler. MR 1728541 (2001f:22029)

[Zel81] Andrey V. Zelevinsky, *Representations of finite classical groups*, Lecture Notes in Mathematics, vol. 869, Springer-Verlag, Berlin-New York, 1981, A Hopf algebra approach. MR 643482 (83k:20017)